Updates and Counterfactuals

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Abstract

We study the problem of combining *updates*—a special instance of theory change—and *counterfactual conditionals* in propositional knowledge bases. Intuitively, an update means that the world described by the knowledge base has changed. This is opposed to *revisions*—another instance of theory change—where our knowledge about a static world changes. A counterfactual implication is a statement of the form 'If A were the case, then B would also be the case', where the negation of A may be derivable from our current knowledge. We present a decidable logic, called $\mathbf{VCU}^2$, that has both update and counterfactual implication as connectives in the object language. Our update operator is a generalization of operators previously proposed and studied in the literature. We show that our operator satisfies certain postulates set forth for any reasonable update. The logic $\mathbf{VCU}^2$ is an extension of D. K. Lewis' logic **VCU** for counterfactual conditionals. The semantics of $\mathbf{VCU}^2$ is that of a multimodal propositional calculus, and is based on possible worlds. The infamous *Ramsey Rule* becomes a derivation rule in our sound and complete axiomatization. We then show that *Gärdenfors' Triviality Theorem*, about the impossibility to combine theory change and counterfactual conditionals via the Ramsey Rule, does not hold in our logic. It is thus seen that the Triviality Theorem applies only to revision operators, not to updates.

Keywords: Belief revision, updates, conditional logic, hypothetical reasoning, theory change.

'Kangaroos have no tails.'

1 Background

The usual *material implication* of formal logic fails to capture many implicative statements of natural language. Consider for instance the sentence '*if I had some oars, I could row across the river*'. If we find ourselves on the bank of a river, equipped with a boat but no oars, then according to the classical truth-table semantics the above sentence is true, in harmony with our intuition. But then, the truth-table method also claims that the sentence '*if I had some oars, pigs could fly*' is true, which obviously contradicts our intuition (cf. [20]).

Much work in logic has been devoted to *conditionals*, that is, implications other than the material one. (For an overview of the field, see [34].) A *counterfactual conditional*, or counterfactual for short, is a statement of the form '*if A, then B*', where, as the name indicates, the premiss A can contradict the current state of affairs, or our current knowledge thereof.

The application of counterfactuals to knowledge bases lies in their ability to express rules of the form '*If A, then B*', and questions of the form '*What if A*', where the negation of A may be derivable from the knowledge base. Among others, Bonner [5] studies a database query language with a *What if*-capability. Ginsberg [20] describes a large number of AI applications of counterfactuals.

So far, we have looked at one side of the coin. On the other side there is the problem of changing the information represented in a knowledge base according to new facts, including facts that contradict the information already in the knowledge base. For a simple example, suppose that the knowledge base contains the fact that '*Tweety is a bird*' and the rule '*All birds* *can fly*'. Then the newly acquired fact '*Tweety cannot fly*' contradicts the knowledge base. Thus we have to give up something. A radical solution would be to throw out all the old information from the knowledge base, and take the new fact as the new knowledge base. There is though a quite substantial body of work on this problem, both in the fields of AI and databases, as well as in the field of logic. (For some overviews, the reader should turn to [26, 31, 40].) A generic name for the problem is *theory change*. The agreement on theory change is that in the case of contradictions, 'as little as possible' of the old theory should be changed in order to accommodate the new facts, that is, the change should be minimal. In the example above, minimal change could mean that we give up either the fact '*Tweety is a bird*', or the rule '*All birds can fly*', but not both.

The notion of minimal change also plays a role in defining the meaning of counterfactual implications. This is explicitly expressed in the *Ramsey Rule* [35]. The Ramsey Rule is summarized by Stalnaker [38] as follows:

> Accept a proposition of the form '*if A, then B*' in a state of belief [or knowledge] K, if and only if the minimal change of K needed to accept A, also requires accepting B.

Gärdenfors [15, 17] has shown that with certain assumptions on the change operation, such operations are incompatible with the Ramsey Rule, meaning that any logic containing the two is trivial (in a sense to be defined later). In other words, it seems to be impossible to combine the facility to change the knowledge base with the ability to express *What if*-questions, and have counterfactual *If-then*-rules, while keeping the very intuitive Ramsey Rule. This result is known as the *Gärdenfors Triviality Theorem*.

Many solutions have been sought to the dilemma of the theorem. Gärdenfors [15] considers weaker versions of the Ramsey Rule, but concludes that these versions do not satisfy our intuition. A more promising direction, where counterfactual *If-then*-rules are forbidden in the knowledge base, is taken by Levi [29]. By going in this direction we however lose the expressive power of the knowledge base, as well as the power to express *iterated* counterfactuals, that is rules of the form *If (If A, then B), then C*. On the other hand, Rott [36] has tried to forbid iterated counterfactuals while allowing (non-iterated) counterfactuals in the knowledge base, and concluded that the triviality result still stands. The works by Makinson [32] and Cross [8] consider the case where the inference operator is non-monotonic, instead of the classical one, but Makinson shows that the triviality theorem holds for all reasonable inference operators. Finally, Gärdenfors [15] and Arlo Costa [3] show that weakening the assumptions on the change operator does not provide a way out of the dilemma.[1]

As pointed out in [1, 27, 28], there are two fundamentally different ways of changing knowledge. One can regard the new evidence as contributing to our knowledge about the real world (e.g. *the butler is guilty*). On the other hand, the evidence can reflect a change in the real world (e.g. *the master of the house has been murdered*). Changes of the former kind are called *revisions* by Katsuno and Mendelzon [27], whereas the latter type is given the name *updates* in that same paper. [2]

Gärdenfors and his colleagues [2] have set forth a set of rationality postulates for revision operations. Katsuno and Mendelzon have done the same for update operations [27]. Now Gärdenfors' triviality result holds for the case where the change function satisfies certain (weakened) postulates for revision.

[1] For recent developments in belief revision theory and conditional logic, the reader is asked to consult [12, 13, 14], and the references therein.

[2] Keller and Winslett [28] use the terms *knowledge adding* and *change recording*, instead of revision and update.

In this paper we study the problem of combining updates (as opposed to revisions) and counterfactuals via the Ramsey Rule. Our formalism is that of a multimodal propositional logic. Neither operation is taken as more primitive than the other, and both are connectives in the object language (in contrast to [15], where the revision operator appears on the level of models). The interpretation of the update connective is a generalization of a 'possible models approach' previously proposed and studied in the literature. The interpretation of the counterfactual connective is the one associated with D. K. Lewis' logic **VCU** [30][3] (see also [37]). It turns out that the logic characterized by this combined class of interpretations is obtained by adding the Ramsey Rule as a *derivation rule* to an axiomatization of Lewis' logic **VCU**. The resulting logic, called $\mathbf{VCU^2}$, of counterfactuals and updates is decidable. We also show that the update postulates of Katsuno and Mendelzon [27] are theorems or metatheorems in this logic. Finally, we show that the Gärdenfors Triviality Theorem does not hold in $\mathbf{VCU^2}$.

2 The language $\mathcal{L}_{>,\circ}$

Let $\mathcal{L}$ be the language of propositional calculus, i.e. $\mathcal{L} = \{p_i : i \in \omega\} \cup \{\neg, \wedge, \bot, (,)\}$. The symbol $\bot$ is a 0-ary connective, and denotes the constant *false*. The set of all well-formed sentences of $\mathcal{L}$ is defined in the usual way. Parentheses are omitted wherever there is no risk of confusion. Let ψ and ϕ be sentences. We regard the sentences $\psi \vee \phi, \psi \rightarrow \phi, \psi \leftrightarrow \phi$, and $\top$ as abbreviations of the sentences $\neg(\neg\psi \wedge \neg\phi), (\neg\psi) \vee \phi, (\psi \rightarrow \phi) \wedge (\phi \rightarrow \psi)$, and $\neg(\bot)$, respectively. The language $\mathcal{L}_{>,\circ}$ is obtained by adding the two (non-truth-functional) binary connectives $>$ and $\circ$ to $\mathcal{L}$. This induces the set of all well-formed sentences of $\mathcal{L}_{>,\circ}$. When there is no risk of confusion, we shall by a *sentence* mean a well-formed $\mathcal{L}_{>,\circ}$-sentence.

Intuitively, $\psi \circ \phi$ means the 'the result of *updating* ψ with ϕ'. The intuitive meaning of $\psi > \phi$ is 'If ψ *were* true, then ϕ *would* also be true'.

In a knowledge base setting the language can be used as follows: Let KB be a finite set of $\mathcal{L}_{>,\circ}$-sentences. If we want to update the knowledge base with some new information represented by a $\mathcal{L}_{>,\circ}$-sentence ϕ, the new knowledge base will be represented by the sentence $KB' \circ \phi$, where KB' is the conjunction of the sentences in KB.[4] Likewise, a hypothetical query, such as *'would ψ be true, if ϕ were true?'* is evaluated by testing whether KB logically implies $\phi > \psi$.

3 $\mathcal{L}_{>,\circ}$-models

Changing worlds and theories

Usually a KB is *incomplete*, in the sense that the reality it describes allows for several (non-isomorphic) interpretations. That is, the true state of affairs in the real world of interest is only known to be among a finitely describable set of possibilities. We, or a reasoning agent, do not have enough information to determine which one of these possibilities—the technical term is *possible worlds*—is *de facto* the actual world.[5] Now, when a change—as a result of an action or otherwise—occurs in the real world, we must change our description of the world. The change itself happens to, or in, the actual world. But since we are confined to our set of

[3]VCU stands for *variably* strict conditionals with universality. Strict conditionals are of the form *necessarily* p implies q. Universality means that all worlds are accessible from all other worlds.

[4]Note that nothing (except perhaps bounded computational resources) prevents either the knowledge base, or the sentence ϕ containing the $\circ$ and $>$ connectives in an arbitrary (well-formed) combination.

[5]In the words of J. Hintikka [24], 'one has to keep an eye on more than one possible world'.

possibilities, we must make the change come true in all of our candidate worlds. Semantically, we change each one of the possible worlds *'as little as possible'* in order to make the new state of affairs hold. Our new syntactic description of the world of interest should now correctly reflect the outcome of this *set* of changes. The function that maps the old description to the new, is called an *update*.

The approach described above is also known as the 'possible models approach' [40]:

> In updating a theory T with a sentence ϕ, update each model of T separately. For each model $\mathcal{M}$ of T, choose those models of ϕ that are closest to $\mathcal{M}$. The result is then the theory of the union of all separate updates.

The scenario is different in the case where we have gathered some new information of what we know is and is not possible in a static world. Our revised description of the world of interest will then have to take this into account, in that the set it now describes either contains less (*expansion* of knowledge) or more (*contraction* of knowledge) possible worlds. The most intriguing case in this scenario is that of a *revision*: What should become of our description if newly acquired information completely contradicts what we thought were among the possibilities of the truth?

Before going into the formal definitions of models, we shall give a rudimentary account of the interpretations of sentences in $\mathcal{L}_{>,\circ}$.

We consider a set I of possible worlds. If a sentence ϕ is true in a world $i \in I$, we say that i is a ϕ-world. To capture the concept of 'closeness', we will associate a total pre-order $\leq_i$ with each possible world i. Then $j \leq_i k$ means that world j is as 'close' to world i as world k is. [6]

The worlds where a sentence $\phi \circ \psi$ is true, are

$$\bigcup_i \{j \in I : j \text{ is a } \psi\text{-world, and } j \text{ is } \leq_i \text{ minimal}\},$$

where i ranges over all ϕ-worlds.

The meaning of a counterfactual, such as *'If kangaroos had no tails, they would topple over'*, is that in any possible world in which kangaroos have no tails, and which resembles our own as much as possible otherwise, kangaroos would topple over [30]. Thus the worlds where a counterfactual $\phi > \psi$ is true are the worlds where the closest worlds in which ϕ is true also have ψ as true. In the notation used above, the worlds were $\phi > \psi$ is true are

$$\{i \in I : \text{the } \leq_i \text{ minimal } \phi\text{-worlds are also } \psi\text{-worlds}\}.$$

A *model* is then a set I of worlds, a total pre-order $\leq_i$ on I for each $i \in I$, and a *valuation* that assigns a subset J of I to each propositional letter p. Then J is the set of p-worlds. The meaning of the Boolean connectives is the usual one, e.g. the $\phi \wedge \psi$-worlds are the worlds that are both ϕ-worlds and ψ-worlds.

We say that a sentence ϕ is *valid in a model*, if all worlds are ϕ-worlds. A sentence ϕ is *valid (unconditionally)* if ϕ is valid in all models.

We should now be ready for a formal description of the semantic apparatus. We shall give three alternative classes of interpretations: one based on total pre-orders (as above), one based on so called spheres, and one based on so called selection functions. It turns out that all three

[6] A total pre-order is a binary relation that is reflexive, transitive, and connected. Note that antisymmetry is not required. Thus it can very well be the case that j is as 'close' to i as k is, and that k is as 'close' to i as j is, without j and k being the same world.

Some concrete orders between possible worlds that have been proposed in the literature are reviewed in [25].

types of models define the same class of valid sentences (Corollaries 3.1 and 3.2 below). Preorders have mainly been used to model updates (cf. [25, 26, 27]), while selection functions form the main semantic apparatus for interpreting conditionals (cf. [7]). Spheres have been used by Lewis [30] and by Grove [22] to model counterfactuals and revisions, respectively. All three classes of models will also (implicitly or explicitly) have an accessibility relation (as in Kripke models). In the end we shall however be able to disregard the accessibility relations altogether, i.e. all worlds will be accessible from all other worlds. This is achieved through a condition called universality, and its corresponding axiom schemata.

Order models

A $\mathcal{L}_{>,\circ}$ - *order model* $\mathcal{I}$ is a quadruple $< I, \leq, R, [\![]\!] >$, where I is a non-empty set of 'worlds' $i, j, k, \ldots$, and $\leq$ is a function that assigns a total pre-order $\leq_i$ over I to each member i of I, where R is a binary reflexive relation over I, and where $[\![]\!]$ is a function that assigns a subset $[\![\phi]\!]$ of I to each sentence ϕ. The relation $\leq_i$ is a *comparative similarity ordering of worlds w.r.t. world* i. The relation R is an *accessibility relation*. Let us introduce the notational shorthand R_i to stand for the set $\{j \in I : iRj\}$, for each $i \in I$. Then $j \in R_i$ means that j is *accessible* from i. The function $[\![]\!]$ is called a *valuation*. In addition the eight following conditions have to be fulfilled.

(O1) *Centering*. If $j \leq_i i$ then $j = i$.

(O2) *Priority*. If $j \in R_i$ and $k \notin R_i$, then $k \not\leq_i j$.

(O3) *Limit assumption*. For each sentence ϕ, the set $min_{\leq_i}([\![\phi]\!] \cap R_i)$ is non-empty, whenever $[\![\phi]\!] \cap R_i$ is non-empty.[7, 8]

The conditions for the valuation function are the following. [9]

(V1) $[\![\perp]\!] = \emptyset$.

(V2) $[\![\neg\phi]\!] = I \setminus [\![\phi]\!]$.

(V3) $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$.

(V4) $[\![\phi > \psi]\!] = \{i \in I : min_{\leq_i}([\![\phi]\!] \cap R_i) \subseteq [\![\psi]\!]\}$.

(V5) $[\![\phi \circ \psi]\!] = \bigcup_{i \in [\![\phi]\!]} min_{\leq_i}([\![\psi]\!] \cap R_i)$.

A sentence ϕ is *valid in a model* $\mathcal{I} =< I, \leq, R, [\![]\!] >$, if $[\![\phi]\!] = I$. A set Σ of sentences is valid in a model $\mathcal{I}$ if all its members are valid in the model $\mathcal{I}$.

Although a quadruple $< I, \leq, R, [\![]\!] >$ that satisfies conditions (O1) – (O3) and (V1) – (V5) above counts as an order model, there are two further conditions that can be imposed on such a quadruple.

[7] For $J \subseteq I$ and $i \in I$, $min_{\leq_i}(J)$ denotes the set $\{j \in J : \text{if } k \in J \text{ then } j \leq_i k\}$.

[8] Condition (O1) states that no world is as close to a world i as i itself. Condition (O2) states that accessible worlds are closer than inaccessible ones. Condition (O3) guarantees that no set $[\![\phi]\!] \cap R_i$ contains an infinite descending chain $j_1 >_i j_2 >_i j_3 >_i \ldots$, where $j >_i k$, denotes the fact $k \leq_i j$, and $j \not\leq_i k$. A discussion of condition (O3) can be found in [30, 34].

[9] The conditions (V1) – (V3) state the usual Boolean semantics for the truth-functional connectives. Condition (V4) says that i is a $\phi > \psi$-world iff all closest accessible ϕ-worlds also are ψ-worlds. Consequently, a counterfactual $\phi > \psi$ can be *vacuously* true at a world i. This happens when $[\![\phi]\!] \cap R_i = \emptyset$. According to condition (V5), the result of updating ϕ with ψ is the closest accessible ψ-worlds, from the viewpoint of each ϕ-world separately. Thus it can happen that in a particular ϕ-world i, there are no accessible ψ-worlds. If this were true in all ϕ-worlds i, the result of the update would be empty. Vacuous truth of a counterfactual and empty results of updates are, however, prevented through a condition called universality: all worlds are accessible from all other worlds (condition (R2) below).

(R1) *Local uniformity.*[10] If $j \in R_i$, then $R_j = R_i$.
(R2) *Universality.* $R_i = I$, for all $i \in I$.

An order model that satisfies (R1) is called a *locally uniform* order model, and if (R2) is satisfied, it is called a *universal* order model. The condition of interest is (R2), universality. Local uniformity is introduced primarily for technical reasons. It will turn out that the class of all locally uniform order models and the class of all universal order models validate the same set of sentences. We will return to this question shortly (Corollary 3.2).

Sphere models

A $\mathcal{L}_{>,\circ}$- *sphere model* $\mathcal{I}$ is a triple $< I, \$, [\![\,]\!] >$, where I and $[\![\,]\!]$ are as in order models, and where $\$$ is a function that assigns a set $\$_i$ of subsets of I to each member i of I. The function $\$$ is called a *system of spheres*, and the members of each $\$_i$ are called *spheres*. In addition, conditions (S1) –(S3), (V1) – (V3), (V4′) and (V5′) below have to be fulfilled.

(S1) *Centering.* $\{i\} \in \$_i$.
(S2) *Nesting.* For all S and T in $\$_i$, $S \subseteq T$, or $T \subseteq S$.
(S3) *Limit assumption.* If $\bigcup\$_i \cap [\![\phi]\!] \neq \emptyset$, then the set $\{S \in \$_i : S \cap [\![\phi]\!] \neq \emptyset\}$ has a $\subseteq$-smallest member.

Let $i \in I$. Then $[\![\phi]\!]_{\$_i}$ denotes $[\![\phi]\!] \cap S$, where S is the $\subseteq$-smallest sphere in $\$_i$, such that $[\![\phi]\!] \cap S \neq \emptyset$. If no such sphere exists, then $[\![\phi]\!]_{\$_i} = \emptyset$.

(V4′) $[\![\phi > \psi]\!] = \{i \in I : [\![\phi]\!]_{\$_i} \subseteq [\![\psi]\!]\}$.
(V5′) $[\![\phi \circ \psi]\!] = \bigcup_{i \in [\![\phi]\!]} [\![\psi]\!]_{\$_i}$.

Conditions (V1) – (V3) are the same as in order models.[11]

The conditions of local uniformity and universality in sphere models are expressed through the sphere function and take the following forms.

(S4) *Local uniformity.* For all $j \in \bigcup\$_i$, $\bigcup\$_j = \bigcup\$_i$.
(S5) *Universality.* For all $i \in I$, $\bigcup\$_i = I$.

Here the *nomenclature* is *locally uniform* sphere models, resp. *universal* sphere models. We are now in a position to show the following lemma (cf modal logic S5), extending a corresponding result by Lewis [30] to models for sentences with the $\circ$-connective.

LEMMA 3.1
Let ϕ be a sentence. Then ϕ is valid in all locally uniform sphere models if and only if ϕ is valid in all universal sphere models.

PROOF. Since universality implies local uniformity, the only if direction follows. For the if direction, let $\mathcal{I} =< I, \$, [\![\,]\!] >$ be a locally uniform sphere model. Then let $i \in I$. We define a triple $\mathcal{I}^i =< I^i, \$^i, [\![\,]\!]^i >$ as follows. Let $I^i = \bigcup\$_i$. Then let $\$^i_j = \{S \cap I^i : S \in \$_j\}$.

[10] An equivalent definition would be to require that R is transitive and Euclidean.

[11] We see that a system of spheres is a compact description of both the comparative similarity ordering and the accessibility relation. A world j is closer to i than k is, if the innermost sphere in $\$_i$ containing j is itself contained in the innermost sphere containing k. The inaccessible worlds are outside all spheres in $\$_i$. In Theorem 3.1 below, we will show how to formally transform an order model into a sphere model, and vice versa.

Note that for each $j \in I^i$, $\$^i_j$ is obviously equal to $\$_j$. Finally, for each sentence ϕ, let $[\![\phi]\!]^i = [\![\phi]\!] \cap I^i$.

Clearly $\$^i$ is a system of spheres on I^i. Since $\$$ is locally uniform, we have $\bigcup\$^i_j = I^i$, for each $j \in I^i$. Thus $\$^i$ is universal.

Next, we verify (V1) – (V3), (V4′) and (V5′).

(V1). $[\![\bot]\!]^i = [\![\bot]\!] \cap I^i = \emptyset$.

(V2). $[\![\neg\psi]\!]^i = [\![\neg\psi]\!] \cap I^i = (I \setminus [\![\psi]\!]) \cap I^i = I^i \setminus [\![\psi]\!]^i$.

(V3). $[\![\psi \wedge \phi]\!]^i = [\![\psi \wedge \phi]\!] \cap I^i = [\![\psi]\!]^i \cap [\![\phi]\!]^i$.

(V4′). $[\![\psi > \phi]\!]^i = \{j \in I : [\![\psi]\!]_{\$_j} \subseteq [\![\phi]\!]\} \cap I^i = \{j \in I^i : [\![\psi]\!]_{\$_j} \subseteq [\![\phi]\!]\} = \{j \in I^i : [\![\psi]\!]^i_{\$^i_j} \subseteq [\![\phi]\!]^i\}$.

(V5′). $[\![\psi \circ \phi]\!]^i = (\bigcup_{j \in [\![\psi]\!]} [\![\phi]\!]_{\$_j}) \cap I^i = \bigcup_{j \in [\![\psi]\!]} ([\![\phi]\!]_{\$_j} \cap I^i) = \bigcup_{j \in [\![\psi]\!]} [\![\phi]\!]^i_{\$^i_j}$. Let $j \in [\![\psi]\!]$, and suppose that $[\![\phi]\!]^i_{\$^i_j}$ is non-empty. If $k \in [\![\phi]\!]^i_{\$^i_j}$, then $k \in \bigcup\$_j$, and $k \in \bigcup\$_i$. Since $\$$ is locally uniform, we have $\bigcup\$_i = \bigcup\$_k = \bigcup\$_j$, and thus $j \in \bigcup\$_i = I^i$. Consequently $\bigcup_{j \in [\![\psi]\!]} [\![\phi]\!]^i_{\$^i_j} \subseteq \bigcup_{j \in [\![\psi]\!]^i} [\![\phi]\!]^i_{\$^i_j}$. Inclusion in the other direction is obvious.

Starting from an arbitrary locally uniform sphere model $\mathcal{I} =< I, \$, [\![\,]\!] >$, we have shown, that for each i in I, $\mathcal{I}^i$ is a universal sphere model. Now, let ϕ be a sentence that is valid in all universal sphere models, and let $\mathcal{I}$ be as above. Let $i \in I$. Since $\mathcal{I}^i$ is a universal sphere model, we have $[\![\phi]\!] = I^i$, meaning that $i \in [\![\phi]\!]^i$, i.e. $i \in [\![\phi]\!] \cap I^i$. Thus $i \in [\![\phi]\!]$, and consequently $[\![\phi]\!] = I$. ∎

Selection models

In the literature on counterfactuals, the semantics of the $>$-connective is often given with the use of *selection functions* (see, for example, [7]). We shall do so here also, and we will show the correspondence between order and sphere models and models that are based on selection functions.

Formally, a $\mathcal{L}_{>,\circ}$- *selection model* $\mathcal{I}$ is a quadruple $< I, f, R, [\![\,]\!] >$, where I, R, and $[\![\,]\!]$ are as in order models, and where f is a function that takes a tuple, consisting of a member i of I and a sentence ϕ, to a subset $f_i(\phi)$ of I. In order for $\mathcal{I}$ to count as a selection model, the following conditions have to be satisfied.

(F1) $f_i(\phi) \subseteq [\![\phi]\!] \cap R_i$.
(F2) If $i \in [\![\phi]\!]$, then $f_i(\phi) = \{i\}$.
(F3) If $[\![\psi]\!] \subseteq [\![\phi]\!]$, and $[\![\psi]\!] \cap f_i(\phi) \neq \emptyset$, then $f_i(\psi) = [\![\psi]\!] \cap f_i(\phi)$.
(F4) If $f_i(\phi) = \emptyset$, then $[\![\phi]\!] \cap R_i = \emptyset$.

The function f is called a selection function. [12] Note that the conditions imply that if $[\![\psi]\!] = [\![\phi]\!]$, then $f_i(\psi) = f_i(\phi)$, and that if $[\![\psi]\!] \subseteq [\![\phi]\!]$, and $f_i(\psi) \neq \emptyset$, then $f_i(\phi) \neq \emptyset$.

The requirements for the valuation in a selection model is as in order and sphere models, except that the conditions for the $>$- and $\circ$-connectives are replaced by the following two conditions:

(V4″) $[\![\phi > \psi]\!] = \{i \in I : f_i(\phi) \subseteq [\![\psi]\!]\}$.

[12] Intuitively, the selection function f_i directly picks the elements $min_{\leq_i}([\![\phi]\!] \cap R_i)$.

(V5'') $[\![\phi \circ \psi]\!] = \bigcup_{i \in [\![\phi]\!]} f_i(\psi)$.

The conditions for local uniformity and universality are (R1) and (R2) as in order models. Note that universality implies that if $f_i(\phi) = \emptyset$, then $[\![\phi]\!] = \emptyset$. The meaning of the names *locally uniform* and *universal* selection models should be clear.

Model equivalence

We shall now turn our attention to the correspondence between the three different classes of models. Call two models *equivalent* if they have the same set of worlds, and if the valuation function is the same in both. The following two theorems extend results by Lewis [30] to locally uniform and universal models, and to models of sentences with the $\circ$-connective.

THEOREM 3.2
For any (locally uniform, universal) order model there is an equivalent (locally uniform, universal) sphere model, and vice versa.

PROOF. Let $\mathcal{I} =< I, \$, [\![\,]\!] >$ be a sphere model. The equivalent order model will be $\mathcal{J} =< I, \leq, R, [\![\,]\!] >$, which obviously is equivalent to $\mathcal{I}$, provided that $\mathcal{J}$ indeed is an order model. In order to show this, we shall construct the qualifying function $\leq$ and accessibility relation R from the system of spheres $\$$.

Given $\$_i$ we define $\leq_i$ as follows: $j \leq_i k$ if and only if for all spheres $S \in \$_i$, if $k \in S$, then $j \in S$. Then $\leq_i$ is obviously a total pre-order.

For each $i \in I$, we define $R_i = \cup \$_i$. Since $\{i\} \in \$_i$, we have $i \in R_i$, that is, R is reflexive. Also, if $\mathcal{I}$ is locally uniform or universal, then $\mathcal{J}$ is locally uniform or universal, respectively.

Let us then verify conditions (O1) – (O3).

(O1). If $j \leq_i i$, then $j \in \{i\}$, and consequently $j = i$.

(O2). If $j \in R_i = \cup \$_i$, and $k \leq_i j$, it follows that $k \in R_i$.

(O3). We shall show that $min_{\leq_i}([\![\phi]\!] \cap R_i) = [\![\phi]\!]_{\$_i}$. Then (O3) follows. Let therefore $j \in min_{\leq_i}([\![\phi]\!] \cap R_i)$. This means that $\cup \$_i \cap [\![\phi]\!] \neq \emptyset$. Let S be the $\subseteq$-smallest sphere in $\$_i$, such that $[\![\phi]\!] \cap S \neq \emptyset$. Suppose that $j \notin [\![\phi]\!] \cap S$, and let $k \in [\![\phi]\!] \cap S$. Since $\cup \$_i \cap [\![\phi]\!] \neq \emptyset$, there is a sphere $T \in \$_i$, such that $j \in [\![\phi]\!] \cap T$. If $T \subset S$, we have an obvious contradiction. Therefore it must be that $S \subset T$, which implies that $k \leq_i j$, and $j \not\leq_i k$; a contradiction to the fact that $j \in min_{\leq_i}([\![\phi]\!] \cap R_i)$.

Then, let $j \in [\![\phi]\!]_{\$_i}$. Then there is a sphere S in $\$_i$, such that $[\![\phi]\!]_{\$_i} = [\![\phi]\!] \cap S$. We also have that $j \in [\![\phi]\!] \cap R_i$.

If $min_{\leq_i}([\![\phi]\!] \cap R_i) = \emptyset$, it means that there is an infinite descending chain $j_1 >_i j_2 >_i j_3 >_i \ldots$ of elements in $[\![\phi]\!] \cap R_i$. It then follows that there is an infinite descending chain $S_1 \supset S_2 \supset S_3 \supset \ldots$ of spheres in $\$_i$, such that each sphere in this chain has a non-empty intersection with $[\![\phi]\!]$; a contradiction to (S3).

If $k \in min_{\leq_i}([\![\phi]\!] \cap R_i)$, then $k \leq_i j$. If $j \not\leq_i k$, it means that there is a sphere $T \in \$_i$, such that $k \in T$, $[\![\phi]\!] \cap T \neq \emptyset$, and $T \subset S$; a contradiction. Thus we have that $j \in min_{\leq_i}([\![\phi]\!] \cap R_i)$.

The conditions (V1) – (V3) are not affected by our transformation. Conditions (V4) and (V5) follows from the observation in (O3) above.

Conversely, let $\mathcal{J} =< I, \leq, R, [\![\,]\!] >$ be an order model. Define $\mathcal{I} = < I, \$, [\![\,]\!] >$. We have to show the construction of $\$$. To this purpose we say that j and k in I are *equivalent w.r.t.* i, if and only if $j \leq_i k$ and $k \leq_i j$. Then *equivalent w.r.t.* i is an equivalence relation over I. For each $j \in I$, let j^i denote the corresponding equivalence class of j. Then we define

$$S_i^j = \bigcup_{k \leq_i j} k^i.$$

Finally, let

$$\$_i = \{S_i^j : j \in R_i\}.$$

We have to verify that \$ is a system of spheres.

(S1). Now $i \in S_i^i$. If $j \in S_i^i$, then $j \leq_i i$, and thus, by (O1), $j = i$. Thus $\{i\} = S_i^i \in \$_i$.

(S2). Let S_i^j and S_i^k be two spheres in $\$_i$. It then follows directly from the definition of $\$_i$ that, if $j \leq_i k$, then $S_i^j \subseteq S_i^k$, and if $k \leq_i j$, then $S_i^k \subseteq S_i^j$.

(S3). From (O2) it follows that $\cup\$_i = R_i$. So, if $\cup\$_i \cap [\![\phi]\!] \neq \emptyset$, then $[\![\phi]\!] \cap R_i \neq \emptyset$. By (O3), the set $min_{\leq_i}([\![\phi]\!] \cap R_i)$ is non-empty. Obviously $min_{\leq_i}([\![\phi]\!] \cap R_i) = j^i \cap [\![\phi]\!]$, where $j \in min_{\leq_i}([\![\phi]\!] \cap R_i)$. Since $S_i^j \supseteq j^i$, we have $S_i^j \cap [\![\phi]\!] \neq \emptyset$. Suppose then that there is a sphere $S_i^k \in \$_i$, such that $S_i^k \subset S_i^j$, and $S_i^k \cap [\![\phi]\!] \neq \emptyset$. But this means that $k \leq_i j$, $j \not\leq_i k$, and $k \in [\![\phi]\!] \cap R_i$; a contradiction to the fact that $j \in min_{\leq_i}([\![\phi]\!] \cap R_i)$. Thus S_i^j is the $\subseteq$-smallest sphere in $\$_i$ that has a non-empty intersection with $[\![\phi]\!]$.

The remaining task for showing that $< I, \$, [\![]\!] >$ is a sphere model of the desired kind is to verify (V1) – (V3), (V4'), (V5'), and (S4) and (S5).

Since $\cup\$_i = R_i$, local uniformity (S4) and universality (S5) are preserved by our transformation. The conditions (V1) – (V3) are not affected. In order to verify (V4') and (V5') it suffices to show that $[\![\phi]\!]_{\$_i} = min_{\leq_i}([\![\phi]\!] \cap R_i)$. In (S3) above, we showed that $min_{\leq_i}([\![\phi]\!] \cap R_i) \subseteq [\![\phi]\!]_{\$_i}$. For inclusion in the other direction, take the sphere S_i^j from the construction in (S3) above. We shall show that $S_i^j \cap [\![\phi]\!] \subseteq j^i \cap [\![\phi]\!]$. Let therefore $k \in S_i^j \cap [\![\phi]\!]$. If $k \notin j^i \cap [\![\phi]\!]$, it means that $k \leq_i j$, $j \not\leq_i k$, and consequently that $S_i^k \subset S_i^j$, and $S_i^k \cap [\![\phi]\!] \neq \emptyset$, which is a contradiction to the fact that $[\![\phi]\!]_{\$_i} = S_i^j \cap [\![\phi]\!]$. ∎

Thus we have established a one to one correspondence between order models and sphere models. The next theorem shows that sphere and selection models can also be put in the same correspondence.

THEOREM 3.3
For any (locally uniform, universal) sphere model there is an equivalent (locally uniform, universal) selection model, and vice versa.

PROOF. Let $\mathcal{I} =< I, \$, [\![]\!] >$ be a sphere model. Our desired equivalent selection model will be $\mathcal{J} =< I, f, R, [\![]\!] >$. As in the preceding theorem, we will construct the selection function f and accessibility relation R from the system of spheres \$.

For each $i \in I$, let $R_i = \cup\$_i$, and for each sentence ψ, let $f_i(\psi) = [\![\psi]\!]_{\$_i}$. Let us now verify (F1) – (F4). The first two conditions are obvious. Note that $f_i(\psi) \neq \emptyset$ iff $\cup\$_i \cap [\![\psi]\!] \neq \emptyset$. Then (F4) is immediate. For (F3), let $[\![\psi]\!] \subseteq [\![\phi]\!]$, and let $[\![\psi]\!] \cap f_i(\phi) \neq \emptyset$. Let S be the $\subseteq$-smallest sphere in $\$_i$, such that $S \cap [\![\phi]\!] \neq \emptyset$. By our assumptions, $S \cap [\![\psi]\!] \neq \emptyset$. If there is a $T \in \$_i$, such that $T \cap [\![\psi]\!] \neq \emptyset$, and $T \subset S$, it means that $T \cap [\![\phi]\!] \neq \emptyset$; a contradiction. Thus (F3) follows.

Now $\mathcal{J} =< I, f, [\![]\!] >$ fulfils the conditions for a selection model. We have just shown (F1)–(F4). The conditions (V1)–(V3) are not affected by our transformation. Finally, (V4″) and (V5″) obviously hold, by the definition of f.

If \$ is locally uniform or universal, then, by definition, R is locally uniform or universal, respectively.

For the other direction of the theorem, let $\mathcal{J} =< I, f, R, [\![\,]\!] >$ be a selection model. The desired sphere model is now $\mathcal{I} =< I, \$, [\![\,]\!] >$. Our task is, as above, to construct the suitable system of spheres \$ from the selection function f and the accessibility relation R.

To this purpose, let

$$S_i^{\psi} = \bigcup_{[\![\psi]\!] \subseteq [\![\phi]\!]} f_i(\phi).$$

Then, let $\$_i = \{S_i^{\psi} : \psi \text{ is a sentence}\} \cup \{R_i\}$.

Before we proceed, we shall show that, for any sentences ψ and ϕ,

$$(*) \qquad \text{If } [\![\psi]\!] \cap S_i^{\phi} \neq \emptyset, \text{ then } f_i(\psi) \subseteq S_i^{\phi}.$$

For a proof of (*), suppose that the precondition holds. Then there must be a sentence χ, such that $[\![\phi]\!] \subseteq [\![\chi]\!]$, and $[\![\psi]\!] \cap f_i(\chi) \neq \emptyset$. By the definition of S_i^{ϕ}, we have $f_i(\psi \vee \chi) \subseteq S_i^{\phi}$. Note that $f_i(\psi \vee \chi) \neq \emptyset$.

Suppose now that $[\![\psi]\!] \cap f_i(\psi \vee \chi) = \emptyset$. Then, by (F1), it must be that $f_i(\psi \vee \chi) \subseteq [\![\chi]\!]$, and consequently, by (F3), we have $f_i(\chi) = [\![\chi]\!] \cap f_i(\psi \vee \chi)$. But this is a contradiction to the assumption that $[\![\psi]\!] \cap f_i(\psi \vee \chi) = \emptyset$. Thus (F3) yields $f_i(\psi) = [\![\psi]\!] \cap f_i(\psi \vee \chi)$, and so $f_i(\psi) \subseteq f_i(\psi \vee \chi) \subseteq S_i^{\phi}$, proving (*).

Now, let us look at (S1) – (S3).

(S1). Let ψ be a sentence such that $i \in [\![\psi]\!]$. Then $f_i(\psi) = \{i\}$. If $[\![\psi]\!] \subseteq [\![\phi]\!]$, then $i \in [\![\phi]\!]$, so $f_i(\phi) = \{i\}$. Thus $S_i^{\psi} = \{i\}$.

(S2). From (F1) it follows that R_i includes every sphere in $\$_i$. Then let S_i^{ψ} and S_i^{ϕ} be two spheres in $\$_i$, other that R_i. Assume that $S_i^{\psi} \not\subseteq S_i^{\phi}$, and that $S_i^{\phi} \not\subseteq S_i^{\psi}$. Let $k \in S_i^{\psi}$ and $m \in S_i^{\phi}$, such that $k \notin S_i^{\phi}$ and $m \notin S_i^{\psi}$. By the definition of $\$_i$, there must be sentences μ and ν, such that $[\![\psi]\!] \subseteq [\![\mu]\!], [\![\phi]\!] \subseteq [\![\nu]\!], k \in f_i(\mu) \subseteq S_i^{\psi}$, and $m \in f_i(\nu) \subseteq S_i^{\phi}$. Now, obviously $f_i(\mu \vee \nu) \subseteq S_i^{\psi} \cap S_i^{\phi}$, and $f_i(\mu \vee \nu) \neq \emptyset$. From (*) and our assumption it follows that $[\![\mu]\!] \cap f_i(\mu \vee \nu) = \emptyset$, and that $[\![\nu]\!] \cap f_i(\mu \vee \nu) = \emptyset$. Thus $f_i(\mu \vee \nu) \not\subseteq [\![\mu]\!]$, and $f_i(\mu \vee \nu) \not\subseteq [\![\nu]\!]$, contradicting (F1). Thus it must be that either $k \in S_i^{\phi}$, or $m \in S_i^{\psi}$, or both, meaning that $\$_i$ is nested.

(S3). If $\cup\$_i \cap [\![\psi]\!] \neq \emptyset$, it means that $R_i \cap [\![\psi]\!] \neq \emptyset$, and consequently that $f_i(\psi) \neq \emptyset$, in which case it is obvious that $S_i^{\psi} \cap [\![\psi]\!] \neq \emptyset$. Let T be a sphere in $\$_i$, such that $T \cap [\![\psi]\!] \neq \emptyset$. By (*), we have $f_i(\psi) \subseteq T$. Then, let $j \in S_i^{\psi}$. From the definition of S_i^{ψ}, it follows that there is a sentence ϕ, such that $[\![\psi]\!] \subseteq [\![\phi]\!]$, and $j \in f_i(\phi)$. Now, since $f_i(\psi) \subseteq [\![\phi]\!]$, we have $T \cap [\![\phi]\!] \neq \emptyset$. Thus, by (*), $f_i(\phi) \subseteq T$, and consequently $j \in T$, meaning that $S_i^{\psi} \subseteq T$. That is, S_i^{ψ} is the $\subseteq$-smallest sphere in $\$_i$, that has a non-empty intersection with $[\![\psi]\!]$.

Now we know that $\$_i$ is a system of spheres. The remaining task is to verify (V4') and (V5'). For this, it is sufficient to show that $[\![\psi]\!]_{\$_i} = f_i(\psi)$. Taking into account the elaboration in (S3) above, the missing piece of information is the fact that $S_i^{\psi} \cap [\![\psi]\!] \subseteq f_i(\psi)$. Let $j \in S_i^{\psi} \cap [\![\psi]\!]$. Then there must be a sentence ϕ, such that $[\![\psi]\!] \subseteq [\![\phi]\!]$, and $j \in f_i(\phi)$. Since we have assumed that $[\![\psi]\!] \cap f_i(\phi) \neq \emptyset$, we get, by (F3), $f_i(\psi) = [\![\psi]\!] \cap f_i(\phi)$. Thus $j \in f_i(\psi)$.

Since $\cup\$_i = R_i$, condition (S4) follows directly from (R1), and (S5) follows directly from (R2). ∎

COROLLARY 3.4
Let ϕ be a sentence. Then the following conditions are equivalent:

(i) ϕ is valid in all order models.

(ii) ϕ is valid in all sphere models.

(iii) ϕ is valid in all selection models.

COROLLARY 3.5
Let ϕ be a sentence. Then the following conditions are equivalent:

(i) ϕ is valid in all locally uniform order models.

(ii) ϕ is valid in all universal order models.

(iii) ϕ is valid in all locally uniform sphere models.

(iv) ϕ is valid in all universal sphere models.

(v) ϕ is valid in all locally uniform selection models.

(vi) ϕ is valid in all universal selection models.

4 Axiomatization

Our main interest is a sound and complete axiomatization of the set of sentences that are valid in all universal models. By Corollary 3.2 it is enough to consider sentences that are valid in all locally uniform selection models. Our axiom system consists of the following axioms and derivation rules.

(A1) All truth-functional axioms.
(A2) $\phi > \phi$.
(A3) $(\phi > \neg\phi) \rightarrow (\psi > \neg\phi)$.
(A4) $(\phi > \neg\psi) \vee (((\phi \wedge \psi) > \chi) \leftrightarrow (\phi > (\psi \rightarrow \chi)))$.
(A5) $((\phi \wedge \psi) > \chi) \leftrightarrow ((\psi \wedge \phi) > \chi)$.
(A6) $(\phi > \psi) \rightarrow (\phi \rightarrow \psi)$.
(A7) $(\phi \wedge \psi) \rightarrow (\phi > \psi)$.
(A8) $(\phi > \bot) \rightarrow (\neg(\phi > \bot) > \bot)$.
(A9) $\neg(\phi > \bot) \rightarrow ((\phi > \bot) > \bot)$.

The rules are the following:

(MP) *Modus Ponens.* $(\phi, \phi \rightarrow \psi) \mapsto \psi$.
(CR) *Counterfactual rule.* For any $n \geq 1$,
$((\chi_1 \wedge \ldots \wedge \chi_n) \rightarrow \psi) \mapsto ((\phi > \chi_1 \wedge \ldots \wedge \phi > \chi_n) \rightarrow (\phi > \psi))$.
(RR) *Ramsey's rules.* $(\chi \rightarrow (\phi > \psi)) \mapsto ((\chi \circ \phi) \rightarrow \psi)$,
and $((\chi \circ \phi) \rightarrow \psi) \mapsto (\chi \rightarrow (\phi > \psi))$.

The intuitive interpretation of (RR) is as follows. Let original belief state be χ and let $\phi > \psi$ stand for '*If* ϕ, *then* ψ.' If $\phi > \psi$ is accepted in state χ it means that $\chi \rightarrow (\phi > \psi)$ is a theorem (see [17]). Now 'the minimal change' of χ 'needed to accept' ϕ, which is represented by $\chi \circ \psi$ 'also requires accepting' ψ, since $(\chi \circ \phi) \rightarrow \psi$ is a theorem, according to (RR).

Note that if we omit (RR), we have an axiomatization of Lewis' logic **VCU** for counterfactuals. If we in addition omit (A8) and (A9) we get Lewis' **VC**. We therefore name our logic $\mathbf{VCU^2}$, where the second **U** stands for updates.[13]

The logic $\mathbf{VCU^2}$ is obtained by closing the set of all axioms by the rules. Members in this set are called $\mathcal{L}_{>,\circ}$-*theorems*, or simply *theorems*. If a sentence ϕ is a theorem, we write $\vdash \phi$. It turns out that the logic $\mathbf{VCU^2}$ is characterized by all locally uniform selection models, and therefore also by all universal order, sphere, or selection models. Before we show this, we shall state some useful properties of $\mathbf{VCU^2}$.

LEMMA 4.1
The following sentences are $\mathcal{L}_{>,\circ}$-theorems:

(A10) For any $n \geq 1: ((\phi > \psi_1) \vee \ldots \vee (\phi > \psi_n)) \rightarrow (\phi > (\psi_1 \vee \ldots \vee \psi_n))$.
(A11) For any $n \geq 1$: $((\phi > \psi_1) \wedge \ldots \wedge (\phi > \psi_n)) \leftrightarrow (\phi > (\psi_1 \wedge \ldots \wedge \psi_n))$.
(A12) $(\psi \circ \phi) \rightarrow \phi$.
(A13) $(\phi > \bot) \rightarrow (\psi > \neg\phi)$.

PROOF. (A10). Since $\vdash \psi_i \rightarrow (\psi_1 \vee \ldots \vee \psi_n)$, for all $i \in \{1, \ldots, n\}$, we have, by (CR), $\vdash (\phi > \psi_i) \rightarrow (\phi > (\psi_1 \vee \ldots \vee \psi_n))$, for all concerned i:s. Thus $\vdash (\phi > \psi_1) \vee \ldots \vee (\phi > \psi_n) \rightarrow \phi > (\psi_1 \vee \ldots \vee \psi_n)$.

(A11). By (A1), $\vdash \psi_1 \wedge \ldots \wedge \psi_n \rightarrow \psi_1 \wedge \ldots \wedge \psi_n$, and thus, by (CR), $\vdash (\phi > \psi_1) \wedge \ldots \wedge (\phi > \psi_n) \rightarrow \phi > (\psi_1 \wedge \ldots \wedge \psi_n)$. Since $\vdash \psi_1 \wedge \ldots \wedge \psi_n \rightarrow \psi_i$, for all $i \in \{1, \ldots, n\}$, (CR) yields $\vdash \phi > (\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \phi > \psi_i$.

(A12). By (A2) we have $\vdash \phi > \phi$. Thus $\vdash \psi \rightarrow (\phi > \phi)$, which gives us $\vdash \psi \circ \phi \rightarrow \phi$, by (RR).

(A13). From (CR), we get $(\phi > \bot) \rightarrow (\phi > \neg\phi)$. Then (A3) yields the desired result. ∎

The following 'substitution of equivalents'-properties will be useful is various proofs.

LEMMA 4.2
(i) If $\vdash \psi \leftrightarrow \phi$, then $\vdash (\chi > \psi) \leftrightarrow (\chi > \phi)$.

(ii) If $\vdash \psi \leftrightarrow \phi$, then $\vdash (\psi > \chi) \leftrightarrow (\phi > \chi)$.

PROOF. (i) follows directly from (CR). For (ii), we note that since $\vdash \psi > \psi$, (CR) yields $\vdash \psi > \phi$, and by a symmetrical argument we get $\vdash \phi > \psi$. Using (CR), we get $\vdash (\psi > \chi) \rightarrow (\psi > (\phi \rightarrow \chi))$. Using truth-functional tautologies, we get

$$\begin{aligned} &\vdash (\psi > \chi) \rightarrow \\ &\qquad (\psi > (\phi \rightarrow \chi)) \wedge (\psi > \neg\phi) \vee \\ &\qquad (\psi > (\phi \rightarrow \chi)) \wedge \neg(\psi > \neg\phi). \end{aligned}$$

Let us look at the first of the disjuncts following the implication. This disjunct implies, by substituting equivalent right-hand sides, $\psi > \neg\psi$. Then we get $\psi > \bot$, from (A11) and

[13] Lewis' axiomatization of VC and the stronger logics does not contain (A5). Instead there is an additional derivation rule: If $\vdash \psi \leftrightarrow \phi$, then $\vdash (\psi > \chi) \leftrightarrow (\phi > \chi)$. This rule is, however, derivable in our system, as we show in Lemma 4.2 (ii) below.

Lemma 4.2 (i), and $\phi > \neg\psi$, from (A13). Now we have $\phi > \bot$, and applying (CR), we obtain $\phi > \chi$.

Then to the second disjunct. Now (A4) tells us that

$$\vdash (\neg(\psi > \neg\phi) \wedge (\psi > (\phi \rightarrow \chi))) \rightarrow ((\psi \wedge \phi) > \chi).$$

By (A5), we have $\vdash ((\psi \wedge \phi) > \chi) \rightarrow ((\phi \wedge \psi) > \chi)$. If $\phi > \neg\psi$, then $\phi > \bot$, and consequently $\phi > \chi$. If $\neg(\phi > \neg\psi)$, then, by (A4), $\vdash ((\phi \wedge \psi) > \chi) \rightarrow (\phi > (\psi \rightarrow \chi))$. Since $\vdash \phi > \psi$, we can apply (A11) and (CR) to obtain $\vdash (\phi > (\psi \rightarrow \chi)) \rightarrow (\phi > \chi)$.

In summary, we have shown that $\vdash (\psi > \chi) \rightarrow (\phi > \chi)$, provided that $\vdash \psi \leftrightarrow \phi$. The rest of the proof is obtained by interchanging ψ and ϕ in the deduction above. ∎

Some further useful properties are expressed in the following lemma.

LEMMA 4.3
The following sentences are $\mathcal{L}_{>,\circ}$-theorems:

(A14) $((\phi > \psi) \wedge (\psi > \bot)) \rightarrow (\phi > \bot)$.
(A15) $((\phi_1 > \bot) \wedge (\phi_2 > \bot)) \rightarrow ((\phi_1 \vee \phi_2) > \bot)$.

PROOF. (A14). From (A13) and Lemma 4.2 *(i)*, we get $((\phi > \psi) \wedge (\phi > \neg\psi)) \rightarrow (\phi > \bot)$. It follows from (A13) that $(\psi > \bot) \rightarrow (\phi > \neg\psi)$. Therefore, we have (A14).

(A15). From (A13) we get $(\phi_1 \vee \phi_2) > \neg\phi_1$, and $(\phi_1 \vee \phi_2) > \neg\phi_2$. Then we use (A11) and substitution of equivalents to derive $(\phi_1 \vee \phi_2) > \neg(\phi_1 \vee \phi_2)$. By (A2) we have $(\phi_1 \vee \phi_2) > (\phi_1 \vee \phi_2)$. Then we apply (A11) and substitution of equivalents once more to get the desired result, $(\phi_1 \vee \phi_2) > \bot$. ∎

Then some definitions and terminology: We say that a sentence ψ is *inconsistent*, iff $\vdash \psi \rightarrow \bot$ (or, equivalently, iff $\vdash \neg\psi$). The sentence ψ is *consistent*, if it is not inconsistent. A set Σ of sentences is consistent iff every finite subset of Σ is consistent. Otherwise Σ is inconsistent. A finite set $\{\psi_1, \ldots, \psi_n\}$ is consistent iff it is not the case that $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \bot$.

Now, ψ *entails* ψ, denoted $\psi \vdash \phi$, iff $\vdash \psi \rightarrow \phi$. Let Σ be a set of sentences. Then $\Sigma \vdash \phi$, if there is a finite subset $\{\psi_1, \ldots, \psi_n\}$ of Σ, such that $\psi_1 \wedge \ldots \wedge \psi_n \vdash \phi$.

A *maximal consistent* set of sentences, is a consistent set of sentences not properly included in any other consistent set of sentences. The proof of the following lemma is the standard one.

LINDENBAUM'S LEMMA 4.4
If Σ is a consistent set of sentences, then Σ can be extended to a maximal consistent set of sentences.

In the sequel we will lean on the properties of maximally consistent sets of sentences, such as $\phi \wedge \psi \in \Sigma$ iff $\phi \in \Sigma$ and $\psi \in \Sigma$, and if $\vdash \phi$ then $\phi \in \Sigma$, etc. (see [6]). Since our system includes truth-functional logic, the following theorem obviously holds.

DEDUCTION THEOREM 4.5
Let Σ be a set of sentences, and let ϕ and ψ be sentences. Then $\Sigma \cup \{\phi\} \vdash \psi$ if and only if $\Sigma \vdash \phi \rightarrow \psi$.

5 Soundness

SOUNDNESS THEOREM 5.1
If a sentence is a $\mathcal{L}_{>,\circ}$-theorem, then it is valid in all locally uniform selection models.

PROOF. (A2). $[\![\phi > \phi]\!] = \{i \in I : f_i(\phi) \subseteq [\![\phi]\!]\} = I$.

(A3). If $i \in [\![\phi > \neg\phi]\!]$, then $f_i(\phi) \subseteq [\![\neg\phi]\!]$. Thus it must be the case that $f_i(\phi) = \emptyset$. Then (F4) tells us that $[\![\phi]\!] \cap R_i = \emptyset$. Consequently we have $R_i \subseteq [\![\neg\phi]\!]$, and so, by (F1), $f_i(\psi) \subseteq [\![\neg\phi]\!]$, meaning that $i \in [\![\psi > \neg\phi]\!]$.

(A4). Suppose that $i \notin [\![\phi > \neg\psi]\!]$. This means that $f_i(\phi) \cap [\![\psi]\!] \neq \emptyset$, and since $[\![\phi]\!] \cap f_i(\phi) = f_i(\phi)$, we have the prerequisites for (F3):

$$[\![\phi \wedge \psi]\!] \subseteq [\![\phi]\!], [\![\phi \wedge \psi]\!] \cap f_i(\phi) \neq \emptyset.$$

Thus $f_i(\phi \wedge \psi) = [\![\phi \wedge \psi]\!] \cap f_i(\phi)$, the latter set being equal to $[\![\psi]\!] \cap f_i(\phi)$.

Now we get $f_i(\phi \wedge \psi) \subseteq [\![\chi]\!]$ if and only if $f_i(\phi) \subseteq (I \setminus [\![\psi]\!]) \cup [\![\chi]\!]$: Let $j \in f_i(\phi)$. If $j \in [\![\psi]\!]$, then $j \in [\![\chi]\!]$. If $j \notin [\![\psi]\!]$, then $j \in (I \setminus [\![\psi]\!])$. Conversely, let $j \in f_i(\phi \wedge \psi)$. Then $j \in f_i(\phi)$, and consequently $j \in [\![\chi]\!]$.

(A5). Since $[\![\phi \wedge \psi]\!] = [\![\psi \wedge \phi]\!]$, we have $f_i(\phi \wedge \psi) = f_i(\psi \wedge \phi)$, for all $i \in I$.

(A6). Let $i \in [\![\phi > \psi]\!]$. If $i \notin [\![\phi]\!]$, then $i \in [\![\phi \rightarrow \psi]\!]$. If $i \in [\![\phi]\!]$, then $i \in f_i(\phi)$. Since $f_i(\phi) \subseteq [\![\psi]\!]$, we have $i \in [\![\psi]\!]$.

(A7). Let $i \in [\![\phi \wedge \psi]\!]$. Then $i \in [\![\phi]\!]$, and $i \in [\![\psi]\!]$. Now $f_i(\phi) = \{i\} \subseteq [\![\psi]\!]$.

(A8). Let $i \in [\![\phi > \bot]\!]$. Then $f_i(\phi) = \emptyset$, and $[\![\phi]\!] \cap R_i = \emptyset$. Suppose that $j \in f_i(\neg(\phi > \bot))$. Then $j \in R_i$. If $f_j(\phi) = \emptyset$, then $j \in [\![\phi > \bot]\!]$, a contradiction. If $f_j(\phi) \neq \emptyset$, then $[\![\phi]\!] \cap R_j \neq \emptyset$, a contradiction to (R1). Thus it must be that $f_i(\neg(\phi > \bot)) = \emptyset$, and $i \in [\![(\neg(\phi > \bot) > \bot)]\!]$.

(A9). Let $i \in [\![\neg(\phi > \bot)]\!]$. Then $f_i(\phi) \neq \emptyset$, and consequently $[\![\phi]\!] \cap R_i \neq \emptyset$. If $j \in f_i(\phi > \bot)$, then $j \in [\![\phi > \bot]\!]$, and $j \in R_i$. Now we get $f_j(\phi) = \emptyset$, and thus $[\![\phi]\!] \cap R_j = \emptyset$; a contradiction to (R1). Thus $f_i(\phi > \bot) = \emptyset$, and $i \in [\![(\phi > \bot) > \bot]\!]$.

(CR). Suppose that $[\![(\chi_1 \wedge \ldots \wedge \chi_n) \rightarrow \psi]\!] = I$. Then $([\![\chi_1]\!] \cap \ldots \cap [\![\chi_n]\!]) \subseteq [\![\psi]\!]$. Suppose then that $f_i(\phi) \subseteq [\![\chi_k]\!]$, for all $k \in \{1, \ldots, n\}$. Then $f_i(\phi) \subseteq [\![\chi_1]\!] \cap \ldots \cap [\![\chi_n]\!] \subseteq [\![\psi]\!]$. Thus $i \in [\![\phi > \psi]\!]$, and consequently $[\![(\phi > \chi_1 \wedge \ldots \wedge \phi > \chi_n) \rightarrow \phi > \psi]\!] = I$.

(RR). Suppose that $[\![\chi]\!] \subseteq \{i \in I : f_i(\phi) \subseteq [\![\psi]\!]\}$. Now $[\![\chi \circ \phi]\!] = \bigcup_{i \in [\![\chi]\!]} f_i(\phi) \subseteq [\![\psi]\!]$.

For the second part, suppose that $\bigcup_{i \in [\![\chi]\!]} f_i(\phi) \subseteq [\![\psi]\!]$. Then $[\![\chi]\!] \subseteq \{i \in I : f_i(\phi) \subseteq [\![\psi]\!]\}$. ∎

THEOREM 5.2
Let Σ be the set of all $\mathcal{L}_{>,\circ}$-theorems. Then Σ is consistent.

PROOF. If Σ is inconsistent there is a finite set $\{\phi_1, \ldots, \phi_n\} \in \Sigma$, such that $\vdash (\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \bot$. By the soundness theorem this amounts to $[\![\phi_1 \wedge \ldots \wedge \phi_n]\!] = \emptyset$, in all selection models. Consider the selection model $\mathcal{I} =< \{i\}, f, R, [\![]\!] >$, where $[\![p]\!] = \{i\}$ for every propositional variable p, where $R = \{(i, i)\}$, and where $f_i(\phi) = \{i\}$, for every sentence ϕ, such that $[\![\phi]\!] \neq \emptyset$. Then $[\![\phi]\!] = \{i\}$, for every ϕ that is an axiom. Since the rules preserve validity, it follows that $[\![\phi]\!] = \{i\}$, for every $\phi \in \Sigma$. Thus Σ must be consistent. ∎

6 Completeness

We will show completeness of our logic using the *canonical models* construction. We note thet similar proofs (for conditional logics) have been constructed by Segerberg [37].

COMPLETENESS THEOREM 6.1
If a sentence ϕ is valid in all locally uniform selection models, then ϕ is a $\mathcal{L}_{>,\circ}$-theorem.

PROOF. We define the *canonical model* $\mathcal{I} = < I, f, R, [\![]\!] >$, where

$$I = \{\, i : i \text{ is a maximal consistent set of sentences } \},$$

and where

$$[\![\psi]\!] = \{i \in I : \psi \in i\},$$

for all sentences ψ. Let $i \in I$. Define

$$R_i = \{j \in I : \text{if } \phi \in j, \text{ then } \neg(\phi > \bot) \in i\}.$$

Then define

$$\theta_i(\phi) = \{\psi : \phi > \psi \in i\}.$$

Note that, by (CR), $\theta_i(\phi)$ is closed under entailment. Furthermore, let

$$\Theta_i(\phi) = \{j \in I : j \text{ is a maximal consistent extension of } \theta_i(\phi)\}.$$

Finally, let

$$f_i(\phi) = \Theta_i(\phi) \cap R_i.$$

Next we verify that $\mathcal{I}$ is a locally uniform selection model. The first point is that I is non-empty. By Theorem 5.2, the set of all $\mathcal{L}_{>,\circ}$-theorems is consistent. Thus this set can be extended to a maximally consistent set. To show that R is reflexive, suppose to the contrary that $i \notin R_i$. Then there must be a sentence ψ, such that $\psi \in i$, and $\psi > \bot \in i$. From (A6), it follows that $\neg\psi \in i$; a contradiction, since i is consistent.

The remaining task is to verify (F1)–(F4), (R1), (V1)–(V3), and (V4″) and (V5″).

(V1). The sentence $\vdash \bot \to \bot$ is a theorem. Thus $\bot$ is inconsistent, i.e. $\bot \notin i$, for all $i \in I$.

(V2). We have $i \in [\![\neg\psi]\!]$ iff $\neg\psi \in i$ iff $\psi \notin i$ iff $i \notin [\![\psi]\!]$ iff $i \in I \setminus [\![\psi]\!]$.

(V3). Now $i \in [\![\psi \wedge \phi]\!]$, iff $\psi \wedge \phi \in i$, iff $\psi \in i$ and $\phi \in i$, iff $i \in [\![\psi]\!] \cap [\![\phi]\!]$.

(F1). If $\psi > \bot \in i$, then $\theta_i(\psi)$ is inconsistent, and consequently $\Theta_i(\psi) = \emptyset$. Thus (F1) holds. Suppose then that $\neg(\psi > \bot) \in i$. By (A2), $\psi > \psi \in i$. Thus $\psi \in \theta_i(\psi)$, and then $\Theta_i(\psi) \subseteq [\![\psi]\!]$. This means that $f_i(\psi) = \Theta_i(\psi) \cap R_i \subseteq [\![\psi]\!] \cap R_i$.

(F2). If $i \in [\![\psi]\!]$, then $\psi \in i$. Let $\phi \in \theta_i(\psi)$. Then $\psi > \phi \in i$, and, by (A6), $\psi \to \phi \in i$, and thus $\phi \in i$. Conversly, let $\phi \in i$. Now $\psi \wedge \phi \in i$, and from (A7) we get $\psi > \phi \in i$, and thus $\phi \in \theta_i(\psi)$. Consequently $i = \theta_i(\psi)$, and since i is maximally consistent, $\Theta_i(\psi) = \{i\}$. Since the relation R is reflexive, we have $i \in R_i$. Now (F2) follows.

(F3). We shall show that if $[\![\psi]\!] \subseteq [\![\phi]\!]$, and $[\![\psi]\!] \cap f_i(\phi) \neq \emptyset$, then $\Theta_i(\psi) = [\![\psi]\!] \cap \Theta_i(\phi)$. If this is true, then (F3) follows. Suppose therefore that the precondition holds.

First we note that $\theta_i(\psi)$ must be consistent, since otherwise $\psi > \bot \in i$, which is a contradiction to the fact that $\emptyset \neq [\![\psi]\!] \cap f_i(\phi) = [\![\psi]\!] \cap R_i \cap \Theta_i(\phi)$.

We shall then collect a few prerequisites. If $\psi > \neg\phi \in i$, then $\Theta_i(\psi) \subseteq [\![\neg\phi]\!]$; a contradiction, since $\Theta_i(\psi) \subseteq [\![\psi]\!] \subseteq [\![\phi]\!]$. Therefore

$$\neg(\psi > \neg\phi) \in i.$$

If $\theta_i(\psi)$ does not entail ϕ, then $\theta_i(\psi) \cup \{\neg\phi\}$ is consistent, which is a contradiction to the fact that $\Theta_i(\psi) \subseteq [\![\psi]\!] \subseteq [\![\phi]\!]$. Thus

$$\psi > \phi \in i.$$

If $\phi > \neg\psi \in i$, we get a contradiction to the fact that $[\![\psi]\!] \cap \Theta_i(\phi) \neq \emptyset$. Hence

$$\neg(\phi > \neg\psi) \in i.$$

Let χ be a sentence. Now $\psi > \chi \in i$ implies, by (A11), that $\psi > (\phi \wedge \chi) \in i$. Since $\vdash (\phi \wedge \chi) \rightarrow (\phi \rightarrow \chi)$, (CR) yields $\psi > (\phi \rightarrow \chi) \in i$. From (A11) it now follows that $\psi > (\phi \wedge (\phi \rightarrow \chi)) \in i$. Now $\vdash (\phi \wedge (\phi \rightarrow \chi)) \rightarrow \chi$. Then, by (CR), we have $\psi > \chi \in i$. Thus we have that

$$\psi > \chi \in i$$

iff

$$\psi > (\phi \rightarrow \chi) \in i.$$

Obviously $\Theta_i(\psi) = [\![\psi]\!] \cap \Theta_i(\phi)$ if and only if, for all sentences χ it is true that $\theta_i(\psi) \vdash \chi$ iff $\theta_i(\phi) \cup \{\psi\} \vdash \chi$. Now

$$\theta_i(\psi) \vdash \chi$$

iff (since $\theta_i(\psi)$ is closed under entailment)

$$\chi \in \theta_i(\psi)$$

iff

$$\psi > \chi \in i$$

iff

$$\psi > (\phi \rightarrow \chi) \in i$$

iff (by (A4))

$$(\psi \wedge \phi) > \chi \in i$$

iff (by (A5))

$$(\phi \wedge \psi) > \chi \in i$$

iff (by (A4))

$$\phi > (\psi \rightarrow \chi) \in i$$

iff (by definition)

$$\theta_i(\phi) \vdash \psi \rightarrow \chi$$

iff (by the Deduction Theorem)

$$\theta_i(\phi) \cup \{\psi\} \vdash \chi.$$

(F4). Suppose that $f_i(\psi) = \emptyset$. If $\Theta_i(\psi) = \emptyset$, then $\theta_i(\psi)$ is inconsistent. Then $\theta_i(\psi) \vdash \bot$, and thus $\psi > \bot \in i$. Suppose then that $j \in [\![\psi]\!] \cap R_i$. We have $\psi \in j$, and since $j \in R_i$, $\neg(\psi > \bot) \in i$; a contradiction.

Next we show that if $\Theta_i(\psi) \neq \emptyset$, then $\Theta_i(\psi) \cap R_i \neq \emptyset$. Then (F4) follows. We shall construct a maximal consistent set j, such that j is in both $\Theta_i(\psi)$ and R_i. Let

$$\Sigma = \theta_i(\psi) \cup \{\neg\phi : \phi > \bot \in i\}.$$

Let Γ denote the set $\{\neg\phi : \phi > \bot \in i\}$. Suppose that Γ is inconsistent. Then there are sentences $\neg\phi_1, \ldots, \neg\phi_n$ in Γ, such that $\vdash \neg(\neg\phi_1 \wedge \ldots \wedge \neg\phi_n)$, i.e. $\vdash \phi_1 \vee \ldots \vee \phi_n$. Thus at least one of the disjuncts, say ϕ_k, is in i. Since $\phi_k > \bot \in i$, we get from (A6) that $\phi_k \to \bot \in i$, and thus that $\neg\phi_k \in i$. We now have a contradiction. Consequently Γ is consistent.

We assumed that $\theta_i(\psi)$ is consistent. Suppose therefore that $\theta_i(\psi) \cup \Gamma$ is inconsistent. This means that $\theta_i(\psi) \vdash \phi_1 \vee \ldots \vee \phi_n$, where $\{\neg\phi_1, \ldots, \neg\phi_n\} \subseteq \Gamma$. Then $\psi > (\phi_1 \vee \ldots \vee \phi_n) \in i$.

From the definition of Γ it follows that $\phi_k > \bot \in i$, for all $k \in \{1, \ldots, n\}$. Then a repeated application of (A15) derives the fact that $(\phi_1 \vee \ldots \vee \phi_n) > \bot \in i$. Now we can apply (A14) and get a contradiction, namely the fact that $\psi > \bot \in i$, contrary to our assumption that $\theta_i(\psi)$ is consistent. Thus it must be that Σ is consistent.

Next, we extend Σ to a maximal consistent set j. Clearly j is in $\Theta_i(\psi)$. If j is not in R_i, then there is a sentence $\phi \in j$, such that $\phi > \bot \in i$. But this means that $\neg\phi$ is in Γ, and consequently in j, which is impossible.

(R1). Let $j \in R_i$, and $k \in R_j$. If $\phi \in k$, then $\neg(\phi > \bot) \in j$. If $\phi > \bot \in i$, then, by (A8) $\neg(\phi > \bot) > \bot \in i$. Thus $f_i(\neg(\phi > \bot)) = \emptyset$. But $[\![\neg(\phi > \bot)]\!] \cap R_i \neq \emptyset$; a contradiction. Thus it must be that $\neg(\phi > \bot) \in i$, and so $k \in R_i$, and consequently $R_j \subseteq R_i$.

Then, let $k \in R_i$. If $\phi \in k$, then $\neg(\phi > \bot) \in i$, and consequently, by (A9), $(\phi > \bot) > \bot \in i$. If $\phi > \bot \in j$, then $j \notin R_i$. Thus $\neg(\phi > \bot) \in j$, and so $k \in R_j$, meaning that $R_i \subseteq R_j$.

(V4''). We shall show that

$$[\![\psi > \phi]\!] = \{i \in I : \Theta_i(\psi) \cap R_i \subseteq [\![\phi]\!]\}.$$

If $\psi > \phi$ is inconsistent, then $\psi > \phi$ cannot belong to any consistent set of sentences, and thus $[\![\psi > \phi]\!] = \emptyset$. Also, from (A7) it follows that $[\![\psi \wedge \phi]\!] = \emptyset$. Suppose then that there is an $i \in I$, such that $\Theta_i(\psi) \cap R_i \subseteq [\![\phi]\!]$. If $\Theta_i(\psi) = \emptyset$, then $\psi > \bot$ is in i, and by (CR) $\psi > \phi$ is also then in i, which is a contradiction. Thus it must be that $\Theta_i(\psi) \neq \emptyset$, and in connection with (F4) above, we showed that then it must also be that $\Theta_i(\psi) \cap R_i \neq \emptyset$.

Then, let $j \in \Theta_i(\psi) \cap R_i$. Since $\psi \in \theta_i(\psi)$ we have $\psi \in j$, and since $\Theta_i(\psi) \cap R_i \subseteq [\![\phi]\!]$, we have $\phi \in j$. Thus $\psi \wedge \phi \in j$, a contradiction to the fact that $\psi \wedge \phi$ is inconsistent. Now we have that, if $\psi > \phi$ is inconsistent, then $\{i \in I : \Theta_i(\psi) \cap R_i \subseteq [\![\phi]\!]\} = \emptyset$.

Suppose now that $\psi > \phi$ is consistent. Let $i \in [\![\psi > \phi]\!]$, i.e $\psi > \phi \in i$. Then $\phi \in \theta_i(\psi)$, and if $j \in \Theta_i(\psi)$, then $\phi \in j$. Thus $\Theta_i(\psi) \subseteq [\![\phi]\!]$, and consequently $\Theta_i(\psi) \cap R_i \subseteq [\![\phi]\!]$.

Suppose then that $\Theta_i(\psi) \subseteq [\![\phi]\!]$. Then $\theta_i(\psi) \vdash \phi$, since there otherwise is a $j \in \Theta_i(\psi)$, such that $\neg\phi \in j$, which is a contradiction. Since $\theta_i(\psi)$ is closed under entailment, $\phi \in \theta_i(\psi)$, and thus $\psi > \phi \in i$.

Next, we show that if $\Theta_i(\psi) \cap R_i \subseteq [\![\phi]\!]$, then $\psi > \phi \in i$. The first possibility is that $\Theta_i(\psi) \cap R_i = \emptyset$. Then there must be a sentence χ, such that $\theta_i(\psi) \vdash \chi$, and $\chi > \bot \in i$. Thus $\psi > \chi \in i, \chi > \bot \in i$, so (A14) gives $\psi > \bot \in i$, and then (CR) hands us $\psi > \phi \in i$.

The second possibility is that $\emptyset \neq \Theta_i(\psi) \cap R_i \subseteq [\![\phi]\!]$. In this case it must be that $\theta_i(\psi) \cup \{\neg\chi : \chi > \bot \in i\} \vdash \phi$, since we showed in (F4) above that all maximal consistent extensions of the set $\theta_i(\psi) \cup \{\neg\chi : \chi > \bot \in i\}$ are in $\Theta_i(\psi) \cap R_i$. If $\theta_i(\psi) \cup \{\neg\chi : \chi > \bot \in i\}$ is consistent with $\neg\phi$, then $\Theta_i(\psi) \cap R_i$ is not a subset of $[\![\phi]\!]$.

From the entailment it follows that there are sentences $\neg\chi_1, \ldots, \neg\chi_n$, where $\chi_1 > \bot, \ldots, \chi_n > \bot$ are in i, and such that $\theta_i(\psi) \cup \{\neg\chi_1 \wedge \ldots \wedge \neg\chi_n\} \vdash \phi$. By the Deduction Theorem this is equivalent to $\theta_i(\psi) \vdash (\neg\chi_1 \wedge \ldots \wedge \neg\chi_n) \to \phi$. Thus $\psi > (\neg(\chi_1 \vee \ldots \vee \chi_n) \to \phi) \in i$.

As in (F4) above we then have that $(\chi_1 \vee \ldots \vee \chi_n) > \bot \in i$. Then (A13) yields $\psi > \neg(\chi_1 \vee \ldots \vee \chi_n) \in i$. By using (A11) and (CR) we get $\psi > \phi \in i$.

(V5″). We are going to show that

$$[\![\psi \circ \phi]\!] = \bigcup_{i \in [\![\psi]\!]} (\Theta_i(\phi) \cap R_i).$$

Suppose that $\psi \circ \phi$ is inconsistent. Then, as in (V4″), $[\![\psi \circ \phi]\!] = \emptyset$. We then also have that $\vdash \psi \circ \phi \rightarrow \neg\phi$. From (RR) we get $\vdash \psi \rightarrow \phi > \neg\phi$. So, if $\psi \in i$, then $\phi > \neg\phi \in i$. By (A2) $\phi > \phi \in i$, and therefore $\phi \in \theta_i(\phi), \neg\phi \in \theta_i(\phi)$, meaning that $\theta_i(\phi)$ is inconsistent. Thus $\Theta_i(\phi) = \emptyset$.

Suppose then that $\psi \circ \phi$ is consistent. Let $i \in [\![\psi]\!]$, i.e. $\psi \in i$. Now $\vdash \psi \circ \phi \rightarrow \psi \circ \phi$, and from (RR) we get $\vdash \psi \rightarrow \phi > \psi \circ \phi$. Thus $\psi \circ \phi \in \theta_i(\phi)$, and so $\Theta_i(\phi) \subseteq [\![\psi \circ \phi]\!]$. Consequently $\Theta_i(\phi) \cap R_i \subseteq [\![\psi \circ \phi]\!]$.

Conversely, let $j \in [\![\psi \circ \phi]\!]$, i.e. $\psi \circ \phi \in j$. We shall construct a maximal consistent set i, such that $\psi \in i$, $j \in \Theta_i(\phi)$, and $j \in R_i$.

Define

$$\Sigma = \{\psi\} \cup \{\neg(\phi > \neg\chi) : \chi \in j\}.$$

We claim that Σ is consistent. If this is not the case, then there is a $n \in \omega$, such that $\{\chi_1, \ldots, \chi_n\} \subseteq j$, and $\{\psi\} \cup \{\neg(\phi > \neg\chi_k) : 1 \leq k \leq n\}$ is inconsistent. This means that $\vdash \psi \rightarrow (\phi > \neg\chi_1) \vee \ldots \vee (\phi > \neg\chi_n)$. Since (CR) yields $\vdash (\phi > \neg\chi_k) \rightarrow (\phi > \neg(\chi_1 \wedge \ldots \wedge \chi_n))$, for all $k \in \{1, \ldots n\}$, we get $\vdash \psi \rightarrow (\phi > \neg(\chi_1 \wedge \ldots \wedge \chi_n))$. By (RR), we have $\vdash \psi \circ \phi \rightarrow \neg(\chi_1 \wedge \ldots \wedge \chi_n)$, meaning that $\neg(\chi_1 \wedge \ldots \wedge \chi_n) \in j$, which is a contradiction.

Thus Σ is consistent. Next, we extend Σ to a maximal consistent set i. Then $\theta_i(\phi)$ is consistent. Otherwise we have $\theta_i(\phi) \vdash \bot$, and since $\theta_i(\phi)$ is closed under entailment we have $\bot \in \theta_i(\phi)$, and thus $\phi > \bot \in i$. If $\phi > \bot$ is in i, (CR) tells us that $\phi > \neg\phi$ also must be in i. On the other hand, from (A12) it follows that ϕ is in j. Thus $\neg(\phi > \neg\phi)$ is in Σ, and consequently in i, which means that we have a contradiction.

Suppose then that $j \cup \theta_i(\phi)$ is inconsistent. Then there is a finite subset $\{\chi_1, \ldots, \chi_n\}$ of j, and a finite subset θ' of $\theta_i(\phi)$, such that $\vdash \theta' \rightarrow \neg(\chi_1 \wedge \ldots \wedge \chi_n)$, and, as above, $\neg(\chi_1 \wedge \ldots \wedge \chi_n) \in \theta_i(\phi)$, meaning that $\phi > \neg(\chi_1 \wedge \ldots \wedge \chi_n) \in i$, which obviously is a contradiction.

Since $j \cup \theta_i(\phi)$ is consistent, and j is a maximal consistent set, we have $\theta_i(\phi) \subseteq j$, which means that $j \in \Theta_i(\phi)$.

It remains to show that $j \in R_i$. If this is not the case, then there is a sentence $\chi \in j$, such that $\chi > \bot \in i$. From (A13) we then get that $\phi > \neg\chi$ must also be in i, which is a contradiction, since i is an extension of Σ.

We have thus verified that $\mathcal{I}$ is indeed a locally uniform selection model. Now we have that ϕ is a $\mathcal{L}_{>,\circ}$-theorem iff ϕ is valid in $\mathcal{I}$: If ϕ is a $\mathcal{L}_{>,\circ}$-theorem, then ϕ belongs to every maximal consistent set of sentences. Thus ϕ is valid in $\mathcal{I}$.

If ϕ is not a $\mathcal{L}_{>,\circ}$-theorem, then $\neg\phi$ is consistent. Thus there is a maximal consistent set $i \in I$, such that $\neg\phi \in i$. Consequently $\phi \notin i$. Thus ϕ is not valid in $\mathcal{I}$. ∎

COROLLARY 6.2

If a sentence ψ is valid in all universal order, sphere, or selection models, then ψ is a $\mathcal{L}_{>,\circ}$-theorem.

7 Decidability

We shall show that our logic has the *small model* property, and that it thus is a decidable logic. First, however, we need some auxiliary definitions and lemmata.

A model $\mathcal{I} = < I, f, R, [\![]\!] >$ is said to be finite, if I is of a finite cardinality, and said to be of cardinality κ, if κ is the cardinality of I.

Let Σ be a set of $\mathcal{L}_{>,0}$-sentences, such that Σ is closed both under truth-functional composition of its members, [14] and under subsentences. We say that a quadruple $\mathcal{I} = < I, f, R, [\![]\!] >$ is a Σ *pre-model*, if $\mathcal{I}$ is a universal selection model, except that the functions f and $[\![]\!]$ are defined only on members of $I \times \Sigma$ and of Σ, respectively.

LEMMA 7.1
Let $\mathcal{I} = < I, f, R, [\![]\!] >$ be a finite Σ pre-model. Then there is an extension f' of f, and and extension $[\![]\!]'$ of $[\![]\!]$, such that $< I, f', R, [\![]\!]' >$ is a finite universal selection model.

PROOF. We proceed much as in the proof of Theorem 3.1. Define

$$S_i^{\psi} = \bigcup_{\{\phi \in \Sigma : [\![\psi]\!] \subseteq [\![\phi]\!]\}} f_i(\phi).$$

Then, we define $\$_i = \{S_i^{\psi} : \psi \in \Sigma\} \cup \{R_i\}$.

Then, as in Theorem 3.1, we show that if $[\![\psi]\!] \cap S_i^{\phi} \neq \emptyset$, then $f_i(\psi) \subseteq S_i^{\phi}$, except that this time the property will hold only for sentences ψ and ϕ in Σ. Since Σ is closed under truth-functional composition, all sentences used in proving the above property are in Σ.

Then we show that $\$$ is a universal system of spheres.

(S1). Since Σ is closed under truth-functional composition, there will, for all $i \in I$, be a sentence $\psi \in \Sigma$, such that $i \in [\![\psi]\!]$. Then we proceed as in Theorem 3.1.

(S2). We look at the proof of (S2) in Theorem 3.1, and note that the sentences μ, ν, and $\mu \vee \nu$ are sentences in Σ.

(S3). Since I is finite, the limit assumption is immediate.

(S5). Universality is fulfilled, since R is universal.

Thus $\$$ is is a universal system of spheres. Then define $[\![p]\!]'$, for all letters outside Σ. The choice of values at this point is irrelevant for our construction. Let $[\![\psi]\!]' = [\![\psi]\!]$, for all sentences $\psi \in \Sigma$. For the remaining sentences, define $[\![]\!]'$ inductively, using the equations (V1)–(V3), (V4′) and (V5′). As in the proof of Theorem 3.1, we can show that $[\![\psi]\!]'_{\$_i} = f_i(\psi)$, this time, however, only for sentences ψ in Σ. Now $< I, \$, [\![]\!]' >$ is a universal sphere model.

Finally we define

$$f_i'(\psi) = [\![\psi]\!]'_{\$_i},$$

for *all* sentences ψ. Since, by our construction, we have $R_i = \cup \$_i$, it follows from Theorem 3.1 that $< I, f', R, [\![]\!]' >$ is a universal selection model.

The function $[\![]\!]'$ is by definition an extension of $[\![]\!]$. It remains to show that f' is an extension of f, i.e. that $f_i'(\psi) = f_i(\psi)$, for all $i \in I$, and for all $\psi \in \Sigma$.

Since $[\![\psi]\!]'_{\$_i} = f_i(\psi)$, for all $\psi \in \Sigma$, we have that $f_i'(\psi) = [\![\psi]\!]'_{\$_i} = f_i(\psi)$. ∎

THEOREM 7.2
Let ψ be a sentence. Then ψ is valid in all universal selection models, if and only if ψ is valid in all universal selection models of a finite cardinality bounded by a function of the number of subsentences in ψ.

[14] That is, $\perp \in \Sigma$, and if ϕ and ψ are in Σ, then $\neg\phi$ and $\phi \wedge \psi$ are also in Σ.

PROOF. Let $\mathcal{I} =< I, f, R, [\!] >$ be an arbitrary universal selection model, and let ψ be a sentence. We shall construct the *filtration* $\mathcal{I}^*$ *of* $\mathcal{I}$ *through* ψ.

Let $Sub(\psi)$ be the set of all subsentences of ψ (we regard ψ as a subsentence of itself). Then, let Σ be the closure of $Sub(\psi)$ under $\wedge$ and $\neg$. Also, let

$$\Gamma = \Sigma \cup \{\chi > \neg\phi : \chi \in \Sigma, \phi \in \Sigma\}.$$

We use the name Δ for the set $\{\chi > \neg\phi : \chi \in \Sigma, \phi \in \Sigma\}$.

We define a relation $\sim$ on $I \times I$, where $i \sim j$ iff for all $\phi \in \Gamma$, $i \in [\![\phi]\!]$ iff $j \in [\![\phi]\!]$. Then $\sim$ is an equivalence relation on I. For each $i \in I$, let i^* denote a unique representative of the equivalence class of i.

Now, consider the quadruple $< I^*, f^*, R^*, [\!]^* >$, where $I^* = \{i^* : i \in I\}$, and the three last component are to be defined shortly. Then we can verify that the cardinality of I^* is less than or equal to $2^{n+2^{2^{n+1}}}$, where n is the number of subsentences in ψ. [15]

Then we define $R_i^* = R_i \cap I^*$. Obviously, R^* is universal (wrt I^*).

Recall at this point that Σ is a subset of Γ. Then, for each $i \in I^*$, and for each $\phi \in \Sigma$, let

$$f_i^*(\phi) = \bigcup_{j \sim i} (f_j(\phi) \cap I^*).$$

The valuation $[\!]^*$ is as follows: For each propositional letter $p \in \Sigma$, let $[\![p]\!]^* = [\![p]\!] \cap I^*$. Let also $[\![\bot]\!]^* = [\![\bot]\!] \cap I^*$. For compound sentences $\phi \in \Sigma$, $[\![\phi]\!]^*$ is defined inductively through equations (V1)–(V3), (V4″) and (V5″).

We are going to show that $< I^*, f^*, R^*, [\!]^* >$ is a Σ pre-model.

We say that a sentence $\phi \in \Sigma$ is *invariant* (w.r.t. *) iff

$$[\![\phi]\!]^* = [\![\phi]\!] \cap I^*.$$

We will show that *all* sentences in Σ are invariant. The proof is by a structural induction.

For the basis, we note that all propositional variables, and the constant $\bot$, are invariant, by definition. For the induction step, let ϕ and χ be invariant sentences in Σ.

(V2). $[\![\neg\phi]\!]^* = I^* \setminus [\![\phi]\!]^* = I^* \setminus ([\![\phi]\!] \cap I^*) = (I \setminus [\![\phi]\!]) \cap I^*$.

(V3). $[\![\phi \wedge \chi]\!]^* = [\![\phi]\!]^* \cap [\![\chi]\!]^* = ([\![\phi]\!] \cap I^*) \cap ([\![\chi]\!] \cap I^*) = ([\![\phi]\!] \cap [\![\chi]\!]) \cap I^* = [\![\phi \wedge \chi]\!] \cap I^*$.

[15] Let $card(A)$ denote the cardinality of the set A. Let Ω be a set of sentences, such that it contains a finite number n of classes of equivalent sentences (two sentences ϕ and ψ are equivalent, if $\vdash \phi \leftrightarrow \psi$). Let $\mathcal{I} =< I, f, R, [\!] >$ be a selection model. Then we define a binary relation $\sim_\Omega$ over I as follows:

$$i \sim_\Omega j \text{ if and only if for all } \phi \in \Omega,\ i \in [\![\phi]\!] \text{ iff } j \in [\![\phi]\!].$$

Then $\sim_\Omega$ is an equivalence relation over I. Let I^Ω be the quotient set of I induced by $\sim_\Omega$. Then it is easily verified that the cardinality of I^Ω is less than, or equal to 2^n. Also, if Ω' is Ω closed under $\wedge$ and $\neg$, then the $card(I^{\Omega'}) = card(I^\Omega)$.

Furthermore, let $\{\Omega_1, \Omega_2\}$ be a partition of Ω. Then it is also easily verified that

$$card(I^\Omega) = card(I^{\Omega_1}) \times card(I^{\Omega_2}).$$

Now the relation $\sim$ defined in the text above is the same relation as $\sim_\Gamma$. Let $n = card(Sub(\psi))$. Then we note that $card(I^{Sub(\psi)}) = card(I^\Sigma) \leq 2^n$. Now, the number of non-equivalent sentences in Σ is bounded by 2^{2^n} (the cardinality of the powerset of I^Σ). Then we use the substitution of equivalents property (Lemma 4.2) to get the fact that the number of non-equivalent sentences in Δ is less than or equal to $2^{2^{n+1}}$. Thus $card(I^\Delta) \leq 2^{2^{2^{n+1}}}$. Finally, we get that $card(I^*) = card(\Gamma) = card(I^{\Sigma\cup\Delta}) = card(I^\Sigma) \times card(I^\Delta) \leq 2^{n+2^{2^{n+1}}}$

(V4″). We have to show that

$$\{i \in I^* : f_i^*(\phi) \subseteq [\![\chi]\!]^*\} = \{i \in I : f_i(\phi) \subseteq [\![\chi]\!]\} \cap I^*.$$

Let $i \in I^*$, such that $f_i^*(\phi) \subseteq [\![\chi]\!]^*$. By the definition of f^* we have that $f_i(\phi) \cap I^* \subseteq [\![\chi]\!] \cap I^*$. Then, let $j \in f_i(\phi)$. If $j = j^*$, then $j \in [\![\chi]\!]$. If $j \neq j^*$, then $j^* \in f_i(\phi) \cap I^*$, and consequently $j^* \in [\![\chi]\!]$. Since $j \sim j^*$, and $\chi \in \Gamma$, we have $j \in [\![\chi]\!]$. Thus $f_i(\phi) \subseteq [\![\chi]\!]$, and therefore $\{i \in I^* : f_i^*(\phi) \subseteq [\![\chi]\!]^*\} \subseteq \{i \in I : f_i(\phi) \subseteq [\![\chi]\!]\} \cap I^*$.

For inclusion in the other direction, let $i \in I^*$, such that $f_i(\phi) \subseteq [\![\chi]\!]$. Since $\phi > \chi \in \Gamma$, we have that if $j \sim i$, then $f_j(\phi) \subseteq [\![\chi]\!]$. Consequently $f_j(\phi) \cap I^* \subseteq [\![\chi]\!] \cap I^*$. Now we have that $f_i^*(\phi) \subseteq [\![\chi]\!]^*$.

(V5″). We must show that

$$\bigcup_{i \in [\![\phi]\!]^*} f_i^*(\chi) = (\bigcup_{i \in [\![\phi]\!]} f_i(\chi)) \cap I^*.$$

Let $i \in [\![\phi]\!] \cap I^*$, and $k \in f_i^*(\chi)$. Then $k \in f_j(\chi) \cap I^*$, for some $j \sim i$. Since $\phi \in \Gamma$, we have $j \in [\![\phi]\!]$. Consequently $k \in \bigcup_{i \in [\![\phi]\!]} (f_i(\chi) \cap I^*) = (\bigcup_{i \in [\![\phi]\!]} f_i(\chi)) \cap I^*$.

To show inclusion in the other direction, let $i \in [\![\phi]\!]$, and $k \in f_i(\chi) \cap I^*$. Since $i \sim i^*$, $k \in f_{i^*}^*(\chi)$. Since $\phi \in \Gamma$, we have $i^* \in [\![\phi]\!]$. By definition, we have $i^* \in I^*$.

Thus all sentences in Σ are invariant. It remains to show that f^* satisfies (F1) – (F4), for all sentences in Σ.

(F1). Since R^* is universal and all sentences ϕ in Σ are invariant, it is enough to show that $f_i^*(\phi) \subseteq [\![\phi]\!] \cap I^*$, for all i in I^*. Let $k \in f_i^*(\phi)$. Then $k \in f_j(\phi) \cap I^*$, for some $j \sim i$. Since f satisfies (F1), we get $k \in [\![\phi]\!] \cap I^*$.

(F2). If $i \in [\![\phi]\!]^*$, then $i \in [\![\phi]\!]$, and so $f_i(\phi) = \{i\}$. If $j \sim i$, then $j \in [\![\phi]\!]$, and thus $f_j(\phi) = \{j\}$. Consequently $f_i^*(\phi) = \{i\}$.

(F3). Suppose that $[\![\phi]\!]^* \subseteq [\![\chi]\!]^*$, and let $k \in [\![\phi]\!]$. Since $\phi \in \Sigma \subseteq \Gamma$, we have $k^* \in [\![\phi]\!]^*$, and thus $k^* \in [\![\chi]\!]^*$. Now χ is also a member of Γ, so we get $k \in [\![\chi]\!]$. Thus $[\![\phi]\!] \subseteq [\![\chi]\!]$.

Suppose then that $[\![\phi]\!]^* \cap f_i^*(\chi) \neq \emptyset$. This is the same as saying that $([\![\phi]\!] \cap I^*) \cap (f_k(\chi) \cap I^*) \neq \emptyset$, for some $k \sim i$, which implies that $[\![\phi]\!] \cap f_k(\chi) \neq \emptyset$. Thus $f_k(\phi) = [\![\phi]\!] \cap f_k(\chi)$.

Then let $j \sim k$, and suppose that $[\![\phi]\!] \cap f_j(\chi) = \emptyset$. This means that $j \in [\![\chi > \neg\phi]\!]$. We also have that $k \notin [\![\chi > \neg\phi]\!]$. Now ϕ and χ are sentences in Σ. Thus $\chi > \neg\phi \in \Gamma$, which contradicts the fact that $j \sim k$. Consequently it must be that $[\![\phi]\!] \cap f_j(\chi) \neq \emptyset$, for all j, such that $j \sim i$. We then have that $f_j(\phi) = [\![\phi]\!] \cap f_j(\chi)$, whenever $j \sim i$. Thus

$$\begin{aligned} & f_i^*(\phi) \\ &= \bigcup_{j \sim i} (f_j(\phi) \cap I^*) \\ &= \bigcup_{j \sim i} (([\![\phi]\!] \cap f_j(\chi)) \cap I^*) \\ &= [\![\phi]\!] \cap (\bigcup_{j \sim i} (f_j(\chi) \cap I^*)) \\ &= [\![\phi]\!] \cap f_i^*(\chi). \end{aligned}$$

(F4). If $f_i^*(\phi) = \emptyset$, we have $f_i(\phi) = \emptyset$. This means that $[\![\phi]\!] \cap R_i = \emptyset$, and consequently that $[\![\phi]\!]^* \cap R_i^* = \emptyset$.

Thus the quadruple $< I^*, f^*, R^*, [\![]\!] >$ is a Σ pre-model. Now Lemma 7.1 tells us that f^* and $[\![]\!]^*$ can be extended to all sentences. Let $f^{*'}$ and $[\![]\!]^{*'}$ denote these extensions. Then $\mathcal{I}^* =< I^*, f^{*'}, R^*, [\![]\!]^{*'} >$ is a universal selection model. Note that, for each $\phi \in \Sigma$, we have $f_i^{*'}(\phi) = f_i^*(\phi)$, and $[\![\phi]\!]^{*'} = [\![\phi]\!]^*$. Thus all sentences in Σ remain invariant when f^* is replaced with $f^{*'}$, and $[\![]\!]^*$ with $[\![]\!]^{*'}$.

In particular, the sentence ψ, that served as the filter, is invariant. If ψ is valid in all universal selection models, then ψ is valid in all finite universal selection models. Suppose then that ψ is not valid in all universal selection models, and let $\mathcal{I} =< I, f, R, [\![]\!] >$, be such a model, where $[\![\psi]\!] \neq I$. Then there is a $i \in I$, such that $i \notin [\![\psi]\!]$. Let $\mathcal{I}^*$ be the filtration of $\mathcal{I}$ through ψ, extended to a universal selection model. Consequently $i^* \notin [\![\psi]\!] \cap I^*$. This means that ψ is not valid in $\mathcal{I}^*$. Furthermore, the cardinality of $\mathcal{I}^*$ is less than or equal to $2^{n+2^{2^n+1}}$, where n is the number of subsentences in ψ. ∎

THEOREM 7.3
There is an algorithm that, given a sentence ψ as input, returns '*yes*', if ψ is a $\mathcal{L}_{>,\circ}$-theorem, and '*no*', otherwise.

PROOF. Let I and J be two sets of the same cardinality. Then it is quite obvious that for every universal selection model $< J, f, R, [\![]\!] >$, and bijection $h : J \rightarrow I$, there is a universal selection model $< I, f', R', [\![]\!]' >$, such that h is an isomorphism, i.e. $h([\![\phi]\!]) = [\![\phi]\!]'$ for all sentences ϕ.

Thus the 'content' of the set J in a model is irrelevant.

Then, let Λ be a subset of the set of propositional letters of $\mathcal{L}_{>,\circ}$. Let $< I, f, R, [\![]\!] >$ be a universal selection model, and let $< I, f, R, [\![]\!]' >$ also be a universal selection model, where $[\![p]\!]' = [\![p]\!]$, for all $p \in \Lambda$, and where $[\![p]\!]' = \emptyset$, for all $p \notin \Lambda$. Then we have that $[\![\phi]\!]' = [\![\phi]\!]$, for all sentences ϕ formed only from letters in Λ.

Finally, let ψ be the sentence the theoremhood of which is to be decided. Let n be the number of subsentences in ψ, and let Λ be the set of all propositional letters occurring in ψ. Then choose a set I with $2^{n+2^{2^n+1}}$ members, for instance the set of all naturals less than this number. Let $\mathcal{M}$ be the set of all models $< J, f, R, [\![]\!] >$, where $J \subseteq I$, and where $[\![p]\!] = \emptyset$, for all $p \notin \Lambda$. Obviously there is a finite number of models in $\mathcal{M}$[16].

If ψ is a $\mathcal{L}_{>,\circ}$-theorem, then, by the soundness theorem, ψ is valid in all universal selection models, and in particular in all models in $\mathcal{M}$. If ψ is not a theorem, then by the completeness theorem, there is a universal selection model $\mathcal{I}$, such that ψ is not valid in $\mathcal{I}$. Let $\mathcal{I}^*$ be the filtration of $\mathcal{I}$ through ψ, extended to a universal selection model. From Theorem 7.1 and the observations above, it follows that there is a model in $\mathcal{M}$, that is $\mathcal{I}^*$, in which ψ is not valid. ∎

Since $\mathbf{VCU^2}$ includes propositional calculus, deciding theoremhood is at least as hard as in the propositional case. As a special case, if there are no occurrences of $>$ or $\circ$ in the sentences ϕ and ψ, then the complexity of checking whether the 'flat' sentences of the forms $\phi > \psi$, and $\phi \circ \psi$ are theorems, is the same as for propositional calculus.

THEOREM 7.4
Let ϕ and ψ be $\mathcal{L}$-sentences (propositional sentences). Then the problem of deciding whether

[16] Disregarding the infinitely many selection functions $f_i(\phi) = \emptyset$, where ϕ mentions letters outside Λ.

the sentence $\phi \circ \psi$ is a $\mathcal{L}_{>,\circ}$-theorem, and whether the sentence $\phi > \psi$ is a $\mathcal{L}_{>,\circ}$-theorem, are both in co-NP.

PROOF. By (A12), $\vdash \psi \circ \phi$ only if $\vdash \phi$. Obviously $\vdash \psi \circ \top \leftrightarrow \psi$, and $\vdash \psi \wedge \phi \rightarrow \psi \circ \phi$. Thus $\vdash \psi \circ \phi$ if and only if $\vdash \psi \wedge \phi$.

By (A6), $\vdash \phi > \psi$ only if $\vdash \phi \rightarrow \psi$. Suppose then that $\vdash \phi \rightarrow \psi$. By (A2) we have $\vdash \phi > \phi$, and then, by (CR), we get $\vdash \phi > \psi$. Thus $\vdash \phi > \psi$ if and only if $\vdash \phi \rightarrow \psi$. ∎

Recently Friedman and Halpern [13] have shown that determining the satisfiability of sentences in **VCU** is complete in exponential time.

In [21] the attention is restricted to a particular order model $\mathcal{I} = < I, \leq, [\![\,]\!] >$, where I is set set of all finite subsets of the set $\{p_i : i \in \omega\}$ of all letters, where $[\![p_i]\!] = \{j \in I : p_i \in j\}$, and where $\leq$ is a *partial* order based on the symmetric difference between worlds.[17] It turns out that determining whether a sentence is valid in this particular structure is complete in polynomial space. On the other hand, validity in a *finite* substructure $\mathcal{J}$ of $\mathcal{I}$ can be determined in time polynomial in the cardinality of $\mathcal{J}$ (i.e. data complexity). When the complexity is based on the size of the sentence (i.e. expression complexity) the problem is in [21] shown to be complete in polynomial space.

8 On updates and revisions

Postulations for change operators

As already briefly discussed, there are (at least) two fundamentally different ways of changing a theory. The first interpretation of some new piece of knowledge is that the world has remained the same, it is only our knowledge about it that is becoming more accurate. This class of changes is called called *revisions*. The other interpretation of a new piece of knowledge is that the world has undergone a change, and that we are to adapt our knowledge to the new situation. Such adaptions are called *updates*.

To highlight the distinction between the two change operators, we borrow an example from J. Biskup [4], who studies the problem of updating relational databases containing 'null values': Suppose that a database contains information about parts in the stock of some enterprise, and that the current state of the database is 'there are balls of an unknown colour in the stock'. The change request is to 'insert' the fact that the stock now contains white balls. There are two ways in which the change request can be interpreted. If, for instance, we have inspected the stock and determined that the existing balls are white, then the change should be a revision, and the resulting database should represent the fact that the stock contains only white balls. If, on the other hand, we have received a new shipping of white balls, the change request is obviously an update. Then the resulting database should represent the facts that there are white balls in the stock, and also balls of an unknown colour, possibly white, possibly some other colour.

For other examples of revisions versus updates, see [27, 39].

Another fundamental distinction can be seen in the way different works models theory change. As we shall see, a change operator can either be a meta level mapping, or an object language connective.

Gärdenfors and his colleagues have set forth a set rationality postulates that any reasonable *revision* operator should satisfy [2]. This approach studies revisions as mappings between

[17] Of course, since $\mathcal{I}$ is based on partial orders instead of total pre-orders, the axiomatization is different than the present one, see [21]. Note that axiomatization in [21] is not proved to be complete.

abstract structures called 'belief sets'. In particular, belief sets can be infinite. The postulates are then conditions on the mappings, and as such, formalizations of our rationality.

In the framework described in this paper the formalization of rationality appears on the level of models: there is a set of possible worlds, and a comparative similarity ordering associated with each possible world. This approach is in the spirit of Kripke models for modal logic. The update is then a connective in the object language, and the interpretation of this connective in in terms of our formalized intuition of possible worlds and similarities. In this setting, we obviously restrict ourselves to updating finite (or finitely axiomatizable) theories. We do not however regard this as a serious restriction, since any knowledge base that is to be used in practice is inherently finite.

The two formalizations of change operators are evidently intertwined. Gärdenfors [16] shows how to construct a system of belief set from a selection model (without the condition for the $\circ$-connective).[18]

As we shall see below, the conditions on a change operator become theorems and metatheorems in the logic having change as a connective.

Next we consider the rationality postulates themselves. In order to avoid introducing any new formalisms and notations, we shall use the formulation of Katsuno and Mendelzon [25], who treat a belief set as a sentence ϕ in propositional calculus. The update operator $\circ$ is then a mapping from pairs of propositional sentences to propositional sentences. That is, $\phi \circ \psi$ should be seen as the denotation of some propositional sentence, and the various postulates as required properties of the mapping $\circ$.

Some additional notation is however unavoidable at this point. Let S be a set of sentences containing at least all truth-functional tautologies. Then $\vdash_{\mathbf{S}} \phi$ denotes the fact that $\phi \in \mathbf{S}$. A sentence ϕ is said to be *consistent in* S if and only if it is not the case that $\neg\phi \in \mathbf{S}$. The sentence ϕ is *complete in* S if and only if, for all sentences ψ in the language for S we have either $\phi \rightarrow \psi \in \mathbf{S}$, or $\phi \rightarrow (\neg\psi) \in \mathbf{S}$. Note that consistent (unqualified) means consistent in $\mathbf{VCU^2}$, and that $\vdash \phi$ means the same as $\vdash_{\mathbf{VCU^2}} \phi$.

Then, let **P** be the set of all truth-functional tautologies. The postulates by Gärdenfors *et al.* [2] for the mapping $\circ$ as a *revision* function are then the following:

(R1) $\vdash_{\mathbf{P}} \phi \circ \psi \rightarrow \psi$.

(R2) If $\phi \wedge \psi$ is consistent in **P**, then $\vdash_{\mathbf{P}} (\phi \circ \psi) \leftrightarrow (\phi \wedge \psi)$.

(R3) If ψ is consistent in **P**, then $\phi \circ \psi$ is also consistent in **P**.

(R4) If $\vdash_{\mathbf{P}} \phi \leftrightarrow \psi$ and $\vdash_{\mathbf{P}} \chi \leftrightarrow \nu$, then $\vdash_{\mathbf{P}} (\phi \circ \chi) \leftrightarrow (\psi \circ \nu)$.

(R5) $\vdash_{\mathbf{P}} ((\phi \circ \psi) \wedge \chi) \rightarrow (\phi \circ (\psi \wedge \chi))$.

(R6) If $(\phi \circ \psi) \wedge \chi$ is consistent in **P** then $\vdash_{\mathbf{P}} (\phi \circ (\psi \wedge \chi)) \rightarrow ((\phi \circ \psi) \wedge \chi)$.

For instance, the rationale behind (R1) is that the new sentence ψ should be true in the new knowledge base. Postulate (R4) assures that the result of the revision is independent of the syntax for the knowledge base ϕ and the new sentence ψ. For an intuitive account of the rest of the postulates, see [2].

The crucial postulate is (R2). It is precisely this postulate that is incompatible with the Ramsey Rule, and which is the major distinction between revisions and updates. If our knowledge of the real world is represented by a sentence ϕ and if we gain some additional knowledge ψ, it is of course quite plausible that our new knowledge should be represented by the sentence $\phi \wedge \psi$, in the case $\phi \wedge \psi$ is consistent. But the situation for updates is different. Consider the

[18] See also [22], and the corresponding remarks in [33] and [19].

following example from the domain of artificial intelligence [26] (cf. also [39]): A room has two objects in it, a book and a magazine. Suppose p_1 means that the book is on the floor, and p_2 means that the magazine is on the floor. Let the knowledge base be $(p_1 \vee p_2) \wedge \neg(p_1 \wedge p_2)$, i.e. either the book or the magazine is on the floor, but not both. Now we order a robot to put the book on the floor, that is, our new piece of knowledge is p_1. If this change is to be taken as a revision, then we find that since the knowledge base is consistent with p_1, our new knowledge base will be equivalent to $p_1 \wedge \neg p_2$, i.e. the book is on the floor and the magazine is not.

But the above situation is inadequate. After the robot moves the book to the floor, all we know is that the book is on the floor; why should we conclude that the magazine is not on the floor? Thus an update would have been more appropriate in this situation.

In determining a set of appropriate postulates for an update function, we have seen that (R2) cannot be used. The set of postulates for the mapping $\circ$ as an update functions are then (R1), (R4), (R5) and (U1) – (U5) below [27]:

(U1) If $\vdash_{\mathbf{P}} \phi \rightarrow \psi$, then $\vdash_{\mathbf{P}} (\phi \circ \psi) \leftrightarrow \phi$.

(U2) If both ϕ and ψ are consistent in **P**, then $\phi \circ \psi$ is also consistent in **P**.

(U3) If $\vdash_{\mathbf{P}} (\phi \circ \psi) \rightarrow \chi$ and $\vdash_{\mathbf{P}} (\phi \circ \chi) \rightarrow \psi$, then $\vdash_{\mathbf{P}} (\phi \circ \psi) \leftrightarrow (\phi \circ \chi)$.

(U4) If ϕ is complete in **P**, then $\vdash_{\mathbf{P}} ((\phi \circ \psi) \wedge (\phi \circ \chi)) \rightarrow (\phi \circ (\psi \vee \chi))$.

(U5) $\vdash_{\mathbf{P}} ((\phi \vee \psi) \circ \chi) \leftrightarrow ((\phi \circ \chi) \vee (\psi \circ \chi))$.

For an intuitive account of these postulates, see [27]. That paper also explains the model theoretic differences between updates and revisions. [19]

It turns out that the $\circ$-connective of our logic $\mathbf{VCU^2}$ satisfies the update postulates in the sense that postulates are theorems or metatheorems in $\mathbf{VCU^2}$.

THEOREM 8.1
Let ϕ, ψ, χ, and ν be $\mathcal{L}$- sentences. Then

(R1). $\vdash \phi \circ \psi \rightarrow \psi$.

(R4). If $\vdash \phi \leftrightarrow \psi$ and $\vdash \chi \leftrightarrow \nu$, then $\vdash (\phi \circ \chi) \leftrightarrow (\psi \circ \nu)$.

(R5). $\vdash ((\phi \circ \psi) \wedge \chi) \rightarrow (\phi \circ (\psi \wedge \chi))$.

(U1). If $\vdash \phi \rightarrow \psi$, then $\vdash (\phi \circ \psi) \leftrightarrow \phi$.

(U2). If both ϕ and ψ are consistent, then $\phi \circ \psi$ is also consistent.

(U3). If $\vdash (\phi \circ \psi) \rightarrow \chi$ and $\vdash (\phi \circ \chi) \rightarrow \psi$, then $\vdash (\phi \circ \psi) \leftrightarrow (\phi \circ \chi)$.

(U4).[20] If ϕ is complete in $\mathbf{VCU^2}$, then $\vdash ((\phi \circ \psi) \wedge (\phi \circ \chi)) \rightarrow (\phi \circ (\psi \vee \chi))$.

(U5). $\vdash ((\phi \vee \psi) \circ \chi) \leftrightarrow ((\phi \circ \chi) \vee (\psi \circ \chi))$.

[19] Katsuno and Mendelzon give a model-theoretic characterization of the update postulates in terms of partial pre-orders on possible worlds. Since a total pre-order also is a partial one, our models satisfy the characterization. We are currently axiomatizing a logic of counterfactuals and updates based on partial pre-orders.

The natural question at this point is whether there exists a finite axiomatization of a logic with updates but no counterfactuals. Indeed, it could be possible to replace the axioms (A2)–(A7) with axioms that only include the $\circ$-connective. But in order to verify (U2), we need the effect of axioms (A8) and (A9). It seems that this effect is not achievable without at least a modal operator of necessity. Therefore, we might as well use the $>$-connective, since it is more natural in the present context. Note that the sentence $(\neg\phi) > \bot$ has the same meaning as the sentence $\Box\phi$ in modal logic S5 (modal logic T, in absence of axioms (A8) and (A9)).

[20] In this case we assume that the language is finitary.

PROOF. Note that if $\vdash_{\mathbf{P}} \phi$, then $\vdash \phi$. Also, if ϕ is consistent (in $\mathbf{VCU^2}$), then ϕ is consistent in **P**. In light of our soundness and completeness theorems we are free to replace theoremhood by validity, and vice versa.

In the following, unless otherwise stated, the valuation $[\![]\!]$ refers to an arbitrary universal selection model $< I, f, R, [\![]\!] >$.

(R1). This is our axiom (A12).

(R4). If $\vdash \phi \leftrightarrow \psi$, and $\vdash \chi \leftrightarrow \nu$, then $[\![\phi]\!] = [\![\psi]\!]$, and $[\![\chi]\!] = [\![\nu]\!]$, and consequently $[\![\phi \circ \chi]\!] = [\![\psi \circ \nu]\!]$.

(R5). Let $i \in [\![(\phi \circ \psi) \wedge \chi]\!]$. Then $i \in [\![\phi \circ \psi]\!]$, and $i \in [\![\chi]\!]$. Thus there must be a $j \in [\![\phi]\!]$, such that $i \in f_j(\psi)$. By (F1) we have $i \in [\![\psi]\!]$. Thus $i \in [\![\psi \wedge \chi]\!]$, and from (F3) it then follows that $i \in [\![\phi \circ (\psi \wedge \chi)]\!]$.

(U1). Let $[\![\phi]\!] \subseteq [\![\psi]\!]$. Then, if $i \in [\![\phi]\!]$, we also have $i \in [\![\psi]\!]$, and by (F2) we have $f_i(\psi) = \{i\}$. Thus $[\![\phi \circ \psi]\!] = [\![\phi]\!]$.

(U2). Let ϕ, and ψ be $\mathcal{L}$-sentences that are consistent. Consider the universal sphere model $< I, \$, [\![]\!] >$, where I is the powerset of all sentence letters in $\mathcal{L}_{>,\circ}$, where $\$_i = \{\{i\}, I\}$, for all $i \in I$, and where $[\![p]\!] = \{i \in I : p \in i\}$, for all letters p. For compound sentences $[\![]\!]$ is defined through equations (V1)–(V3), (V4′) and (V5′). Since ϕ and ψ do not contain the $\circ$ or $>$ connectives, it follows that $[\![\phi]\!] \neq \emptyset$, and $[\![\psi]\!] \neq \emptyset$. From universality it then follows that $[\![\phi \circ \psi]\!] \neq \emptyset$. By the soundness theorem $\phi \circ \psi$ is consistent.

(U3). If $[\![\phi \circ \psi]\!] \subseteq [\![\chi]\!]$, and $[\![\phi \circ \chi]\!] \subseteq [\![\psi]\!]$, then $f_i(\psi) \subseteq [\![\chi]\!]$, and $f_i(\chi) \subseteq [\![\psi]\!]$, whenever $i \in [\![\phi]\!]$. From (F3) it then follows that $f_i(\psi \wedge \chi) = [\![\psi]\!] \cap [\![\chi]\!] \cap f_i(\psi)$, and that $f_i(\psi \wedge \chi) = [\![\psi]\!] \cap [\![\chi]\!] \cap f_i(\chi)$. Thus $f_i(\psi) = f_i(\chi)$, for all $i \in [\![\phi]\!]$.

If $[\![\chi]\!]$is empty, then it follows from (U2) that $[\![\psi]\!]$ or $[\![\phi$ is empty. Anyhow, (U3) follows. The case where $[\![\psi]\!]$ is empty is similar. (U4). Since ϕ is complete in $\mathbf{VCU^2}$, we have either $\vdash \phi \rightarrow (\psi > \neg\chi)$ or $\vdash \phi \rightarrow \neg(\psi > \neg\chi)$. In the first case (RR) gives us $\vdash (\phi \circ \psi) \rightarrow \neg\chi$. From truth-functional reasoning we obtain $\vdash \chi \rightarrow \neg(\phi \circ \psi)$. By (A12) we have $\vdash (\phi \circ \chi) \rightarrow \chi$. Thus $\vdash (\phi \circ \chi) \rightarrow \neg(\phi \circ \psi)$, meaning that $(\phi \circ \psi) \wedge (\phi \circ \chi)$ is inconsistent, and consequently, that (U4) holds.

Suppose then that $\vdash \phi \rightarrow \neg(\psi > \neg\chi)$, and that $(\phi \circ \psi) \wedge (\phi \circ \chi)$ is consistent. Let $< I, \$, [\![]\!] >$ be a universal sphere model, and let $k \in [\![(\phi \circ \psi) \wedge (\phi \circ \chi)]\!]$. Since k is also a member of $[\![\phi \circ \chi]\!]$, there must be a $i \in [\![\phi]\!]$, such that $k \in [\![\chi]\!]_{\$_i}$. Let S be the $\subseteq$-smallest sphere in $\$_i$, such that $S \cap [\![\chi]\!] \neq \emptyset$, i.e. $S \cap [\![\chi]\!] = [\![\chi]\!]_{\$_i}$.

Now, if $k \in [\![\psi \vee \chi]\!]_{\$_i}$, then $k \in [\![\phi \circ (\psi \vee \chi)]\!]$. Suppose therefore to the contrary, that $k \notin [\![\psi \vee \chi]\!]_{\$_i}$. Let $T \in \$_i$ be such that $T \cap [\![\psi \vee \chi]\!] = [\![\psi \vee \chi]\!]_{\$_i}$. Obviously $T \subseteq S$. Since $k \notin [\![\psi \vee \chi]\!]_{\$_i}$, it must be that $T \subset S$, and $T \cap [\![\chi]\!] = \emptyset$, and that $T \cap [\![\psi \vee \chi]\!] = T \cap [\![\psi]\!]$. Consequently we have $[\![\psi]\!]_{\$_i} \subseteq [\![\neg\chi]\!]$, i.e. $i \in [\![\psi > \neg\chi]\!]$. We get $i \in [\![\phi \rightarrow (\psi > \neg\chi)]\!]$, which is a contradiction to the assumption that $\vdash \phi \rightarrow \neg(\psi > \neg\chi)$.

(U5).

$$
\begin{aligned}
& [\![(\phi \vee \psi) \circ \chi]\!] \\
&= \bigcup_{i \in [\![\phi \vee \psi]\!]} f_i(\chi) \\
&= \bigcup_{i \in [\![\phi]\!] \cup [\![\psi]\!]} f_i(\chi)
\end{aligned}
$$

$$= \bigcup_{i \in [\phi]} f_i(\chi) \cup \bigcup_{i \in [\psi]} f_i(\chi)$$

$$= [\![\phi \circ \chi]\!] \cup [\![\psi \circ \chi]\!]$$

$$= [\![(\phi \circ \chi) \vee (\psi \circ \chi)]\!].$$

■

Some further properties of VCU²

In addition to the update postulates, there are some validities concerning the $\circ$-connective, that are of interest. The following theorem contains a list of these validities.

THEOREM 8.2
The following $\mathcal{L}_{>,\circ}$-sentences are $\mathcal{L}_{>,\circ}$-theorems.

(i). $(\phi \circ \psi) \rightarrow \psi$.
(ii). $(\phi \wedge \psi) \rightarrow (\phi \circ \psi)$.
(iii). $(\phi \circ \phi) \leftrightarrow \phi$.
(iv). $((\phi \circ \psi) \circ \psi) \leftrightarrow (\phi \circ \psi)$.
(v). $(\top \circ \phi) \leftrightarrow \phi$.
(vi). $(\phi \circ \top) \leftrightarrow \phi$.
(vii). $(\bot \circ \phi) \leftrightarrow \bot$.
(viii). $(\phi \circ \bot) \leftrightarrow \bot$.
(ix). $((\phi \vee \psi) \circ \chi) \leftrightarrow ((\phi \circ \chi) \vee (\psi \circ \chi))$.
(x). $((\phi \wedge \psi) \circ \chi) \rightarrow ((\phi \circ \chi) \wedge (\psi \circ \chi))$.
(xi). $(\phi \circ (\psi \vee \chi)) \rightarrow ((\phi \circ \psi) \vee (\phi \circ \chi))$.
(xii). $((\phi \circ \psi) \wedge (\phi \circ \chi)) \rightarrow (\phi \circ (\psi \wedge \chi))$.

PROOF. As in the proof of the previous theorem, the valuation $[\![\,]\!]$ used below will refer to an arbitrary selection model $< I, f, R, [\![\,]\!] >$, unless otherwise stated.

(ii). If $i \in [\![\phi \wedge \psi]\!]$, then $f_i(\psi) = \{i\}$.

(iii). Axiom (A12) gives the first half, and the second half follows from the fact that if $i \in [\![\phi]\!]$, then $f_i(\phi) = \{i\}$.

(iv). Follows directly from (A12) and (U1).

(v). Since $[\![\phi]\!] \subseteq [\![\top]\!]$, we have $[\![\phi]\!] \subseteq [\![\top \circ \phi]\!]$.

(vi). For all i, we have $f_i(\top) = \{i\}$.

(vii) and (viii) follow (semantically) directly from the definition of the $\circ$-connective.

(x).

$$[\![(\phi \wedge \psi) \circ \chi]\!]$$

$$= \bigcup_{i \in [\phi] \cap [\psi]} f_i(\chi) \subseteq$$

$$= \bigcup_{i \in [\phi]} f_i(\chi) \cap \bigcup_{i \in [\psi]} f_i(\chi)$$

$$= [\![(\phi \circ \chi) \wedge (\psi \circ \chi)]\!].$$

(xi). Consider a sphere model $< I, \$, [] >$. Let $j \in [\psi \vee \chi]_{\$_i}$, and let S be the $\subseteq$-smallest sphere in $\$_i$, such that $S \cap [\psi \vee \chi] \neq \emptyset$. Now j is a member of $S \cap [\psi] \cup [\chi]$. If $j \in [\psi]$, there can be no properly $\subseteq$-smaller sphere T in $\$_i$, such that $T \cap [\psi] \neq \emptyset$, since we then also would have $T \cap [\psi \vee \chi] \neq \emptyset$. Therefore $j \in [\psi]_{\$_i}$. The case where $j \in [\chi]$ responds to the same argument. Thus $j \in [\psi]_{\$_i} \cup [\chi]_{\$_i}$.

(xii). Since $\vdash (\phi \circ \chi) \rightarrow \chi$, (by (A12)), we get $\vdash ((\phi \circ \psi) \wedge (\phi \circ \chi)) \rightarrow ((\phi \circ \psi) \wedge \chi)$. Similarly to the proof of (R5) in Theorem 8.1 we the get $\vdash ((\phi \circ \psi) \wedge \chi) \rightarrow (\phi \circ (\psi \wedge \chi))$. ∎

A non-triviality result

Gärdenfors has shown that any logic with the Ramsey Rule and with a change connective satisfying postulates (R1), (R2) and (R3) is *trivial* in a certain sense [15, 17]. Gärdenfors shows his theorem in the context of belief sets, but we shall rephrase and reprove it in the current formalism. In the following we assume that all logics include the truth functional tautologies. We shall also need the following auxiliary result.

LEMMA 8.3
Let $\mathbf{L}$ be a logic with $\circ$ and $>$ connectives satisfying Ramsey's Rules (RR). Then $\vdash_{\mathbf{L}} \phi \rightarrow \psi$ implies $\vdash_{\mathbf{L}} (\phi \circ \chi) \rightarrow (\psi \circ \chi)$.

PROOF. Let μ be an arbitrary sentence, such that $\vdash_{\mathbf{L}} (\psi \circ \chi) \rightarrow \mu$. By (RR), we have $\vdash_{\mathbf{L}} \psi \rightarrow (\chi > \mu)$. Consequently $\vdash_{\mathbf{L}} \phi \rightarrow (\chi > \mu)$. Applying (RR) again, we get $\vdash_{\mathbf{L}} (\phi \circ \chi) \rightarrow \mu$. ∎

A logic $\mathbf{L}$ is said to be *non-trivial* if there are at least four sentences ϕ, ψ, χ, and μ in the language for $\mathbf{L}$, such that the sentences $\psi \wedge \chi$, $\psi \wedge \mu$, and $\chi \wedge \mu$ are inconsistent in $\mathbf{L}$, and the sentences $\phi \wedge \psi$, $\phi \wedge \chi$, and $\phi \wedge \mu$ are consistent in $\mathbf{L}$. Otherwise the logic $\mathbf{L}$ is *trivial*.

GÄRDENFORS' TRIVIALITY THEOREM 8.4
Let $\mathbf{L}$ be a logic with $\circ$ and $>$ connectives, such that $\mathbf{L}$ satisfies Ramsey's Rules (RR) and postulates (R1), (R2), and (R3). Then $\mathbf{L}$ is trivial.

PROOF. Suppose to the contrary that $\mathbf{L}$ is non-trivial, and let ϕ, ψ, χ, and μ the sentences that verify the non-triviality. From (R1) we get the fact that $\vdash_{\mathbf{L}} ((\phi \circ \psi) \circ (\chi \vee \mu)) \rightarrow (\chi \vee \mu)$. From (R3) it follows that $(\phi \circ \psi) \circ (\chi \vee \mu)$ is consistent in $\mathbf{L}$. Thus $(\phi \circ \psi) \circ (\chi \vee \mu)$ is consistent with χ or with μ in $\mathbf{L}$. Assume that $(\phi \circ \psi) \circ (\chi \vee \mu)$ is consistent with μ in $\mathbf{L}$; the other case can be proved in a parallel way. Since $\phi \wedge \psi$ is consistent in $\mathbf{L}$, (R2) implies that $\vdash_{\mathbf{L}} (\phi \circ \psi) \rightarrow (\phi \wedge (\psi \vee \chi))$. From Lemma 8.1 we get $\vdash_{\mathbf{L}} ((\phi \circ \psi) \circ (\chi \vee \mu)) \rightarrow ((\phi \wedge (\psi \vee \chi)) \circ (\chi \vee \mu))$. Thus $(\phi \wedge (\psi \vee \chi)) \circ (\chi \vee \mu)$ is consistent with μ in $\mathbf{L}$.

On the other hand, from the assumption about the four sentences, it follows that the sentence $(\phi \wedge (\psi \vee \chi)) \wedge (\chi \vee \mu)$ is consistent in $\mathbf{L}$. Thus (R2) implies that $\vdash_{\mathbf{L}} ((\phi \wedge (\psi \vee \chi)) \circ (\chi \vee \mu)) \rightarrow (\phi \wedge (\psi \vee \chi) \wedge (\chi \vee \mu))$. Since $\psi \wedge \mu$ is inconsistent in $\mathbf{L}$, we have $\vdash_{\mathbf{L}} (\phi \wedge (\psi \vee \chi) \wedge (\chi \vee \mu)) \leftrightarrow \phi \wedge \chi$.

Since $(\phi \wedge \chi) \wedge \mu$ is inconsistent in $\mathbf{L}$, it follows that $(\phi \wedge (\psi \vee \chi)) \circ (\chi \vee \mu)$ is inconsistent with μ in $\mathbf{L}$; a contradiction. Thus $\mathbf{L}$ must be trivial. ∎

The proof of Gärdenfors' Triviality Theorem does not apply to $\mathbf{VCU}^2$, since postulate (R2) is not satisfied. We shall also explicitly show that the our logic is non-trivial.

THEOREM 8.5
The logic $\mathbf{VCU}^2$ is non-trivial.

PROOF. Let p_1, p_2, p_3, and p_4 be propositional letters. Let ϕ, ψ, χ, and μ be the sentences $p_1, (p_2 \wedge \neg p_3 \wedge \neg p_4), (p_3 \wedge \neg p_2 \wedge \neg p_4)$, and $(p_4 \wedge \neg p_2 \wedge \neg p_3)$, respectively. It is clear that the sentences $\psi \wedge \chi, \psi \wedge \mu$, and $\chi \wedge \mu$ are inconsistent.

Consider the quadruple $< I, \leq, R, [\![]\!] >$, where I is the powerset of all sentence letters in $\mathcal{L}_{>,\circ}$, and where $[\![p]\!] = \{i \in I : p \in i\}$, for any letter p. Define $j \leq_i k$ iff the cardinality of the set $\{j \setminus i\} \cup \{i \setminus j\}$ is less than or equal to the cardinality of the set $\{k \setminus i\} \cup \{i \setminus k\}$. Then it is easily verified that $\leq_i$ is a total pre-order on I, for each $i \in I$. [21] Let $R = I \times I$, and for compound sentences, define $[\![]\!]$ through equations (V1) – (V5). Then $< I, \leq, R, [\![]\!] >$ is a universal order model.

Now $[\![\phi \wedge \psi]\!] \neq \emptyset$, since for instance $\{p_1, p_2\} \in [\![\phi \wedge \psi]\!]$. Likewise, we have that $[\![\phi \wedge \chi]\!] \neq \emptyset$, and $[\![\phi \wedge \mu]\!] \neq \emptyset$. It now follows from the soundness theorem that $\mathbf{VCU^2}$ is non-trivial. ∎

9 Conclusions

The logic $\mathbf{VCU^2}$ provides a semantics for updating knowledge bases, with the obvious advantage of decidability. The question of how to specify the orders on possible worlds, pertaining to some particular application—an analogue to the *frame axioms*—requires further studies. One possible avenue would be to assign priorities to the sentences in the knowledge base, as in [18] and [11]. Another possibility is to adapt the *model checking* approach of [23]: Instead of a theory, a knowledge base is a finite model, and query evaluation amounts to checking for validity in that model. This direction is pursued in [21].

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