A Short Proof of the Linear Arboricity for Cubic Graphs

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Dedicated to Prof. Satsuo Kawasaki

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Abstract

Akiyama, Exoo and Harary proved in [1] that the linear arboricity for cubic graphs is 2. They did not seek a short proof of this result but derived it in order to illustrate certain proof techniques, so called "necessary subgraphs for cubic graphs." The purpose of this note is to present a short proof for this result by applying Vizing's Theorem on the edge chromatic number.

In a *linear forest,* each component is a path. The *linear arboricity* (or *path chromatic index*) $\Xi(G)$ of a graph G is defined by Harary in [3] as the minimum number of linear forests whose union is G. Note that the Greek letter, capital *Xi,* looks like three paths!

An assignment of colors to the edges of a nonempty graph G so that adjacent edges are colored differently is an *edge coloring* of G (an *n-edge coloring* if n colors are used). The graph G is n-edge colorable if there exists an m-edge coloring of G for some $m \leqq n$. The minimum n for which a graph G is n-edge colorable is its *edge chromatic number* (or *chromatic index*) and is denoted by $\chi'(G)$.

The Theorem *The linear arboricity for a cubic graph G is 2:*

$$\Xi(G)=2$$

Proof By Vizing's Theorem [4], we have the following inequalities;

$$3=\Delta G \leqq \chi'(G) \leqq \Delta G+1=4,$$

where ΔG stands for the maximum degree of G.

We first color all the lines of G with 4 distinct colors, say, a, b, c and d, such that no adjacent lines have the same color. We replace the color of the lines as follows:

The lines colored with a or b are replaced with color 1.

The lines colored with c or d are replaced with color 2.

The subgraph G_1 (or G_2) induced by the lines with color 1 (or 2) has the maximum degree at most two. i. e., $\Delta G_i \leqq 2$, $i=1, 2$.

If neither G_1 nor G_2 contains a cycle, the theorem is true. We now assume that G_1 or G_2 contains a cycle. Our purpose is to show the possibility that we can replace the color of some lines on each monochromatic cycle with the other color so that no monochromatic cycles are left. Let C_1 be a cycle induced by the lines with color 1, and take three successive points on C_1, say v_1, v_2, v_3. We denote the lines, outside of C_1, incident to v_i by e_i, $i=1, 2, 3$, respectively as illustrated in Figure 1.

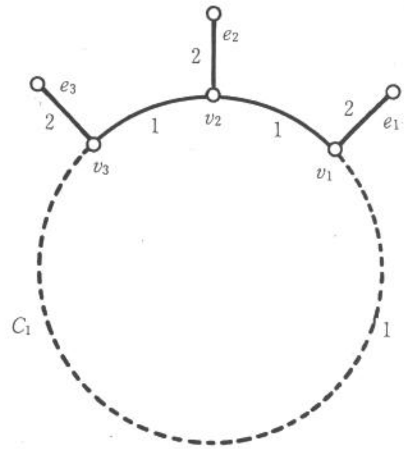

Figure 1

It is obvious that the three lines e_i, $i=1, 2, 3$ have color 2, since $\Delta G_i \leqq 2$ for $i=1,2$.

There are two essentially distinct cases:

CASE 1. *There is no path joining v_2 and v_3, consisting of lines with color 2.* In this case, it is possible to replace the color 1 of the line $\{v_2, v_3\}$ with color 2. As a consequence of the procedure, we avoid a monochromatic cycle C_1 and produce no new monochromatic cycles.

CASE 2. *There is a path P, joining v_2 and v_3, consisting of lines with color 2.* In this case, we show that three are no paths, joining v_2 and v_1, consisting of lines with color 2. Suppose that three exists a path P_1 consisting of lines with color 2 joining v_1 and v_2, see Figure 2.

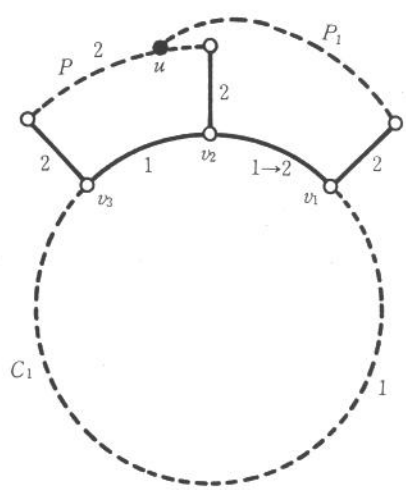

Figure 2

Then three would be a point u on *both* P and P_1, which contradicts the fact that deg $u \leqq 2$ in G_2. Thus we can replace the color 1 of the line $\{v_1, v_2\}$ with color 2 so that no new monochromatic cycles are produced and the monochromatic cycle C_1 is avoided. Repeating the procedure above until all monochromatic cycles are avoided, we complete the proof.

References

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