Paul Erdős and Combinatorics

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It is hard to believe that Paul Erdős has, in his words, "left for the land where one needs no passport". For many of us, 'Uncle Paul' was an integral part of our mathematical lives. A constant traveller, his presence at conferences was taken for granted, almost as an axiom. He referred to lecturing as "preaching", and his lectures were indeed akin to religious ceremonies. Not 'sermons from the Mount' — he was a remarkably unassuming 'preacher' — but rather communions at which this keeper of knowledge past and present (his memory was legendary) probed ever forward, gently but insistently coaxing Mathematics to yield up her secrets through his inexorable questioning. Frequently, a problem raised would appear to the uninitiated somewhat artificial and arbitrary. With experience, however, one realized that it was but one small piece of a large jigsaw puzzle that he was assembling in his mind, and that the full picture would be far more significant than the sum of its parts.

It was in this way that Erdős built or developed, from relatively modest beginnings, substantial and important fields such as Ramsey theory, random graph theory, extremal combinatorics, combinatorial set theory, combinatorial geometry and combinatorial number theory. Monographs on these topics [21, 23, 4, 3, 5, 20, 15, 16, 24] reveal the effectiveness of his 'bottom up' approach to mathematics. Here is a case in point. In 1941, while imprisoned in a labour camp in Hungary, his close friend Paul Turán determined how many edges a graph on n vertices can have without containing a complete subgraph on $m+1$ vertices [27]; Turán proved also that there is a unique extremal graph, known now as the *Turán graph* and denoted $T(m,n)$. Erdős sensed that this fundamental but rather special result was just the tip of the iceberg and, through his constant probing, created the now-flourishing field of extremal graph theory. One of the cornerstones of this theory is the famous Erdős-Stone theorem of 1946, which states, roughly speaking, that every graph on n vertices with more edges than $T(m,n)$ contains not just a complete graph on $m+1$ vertices but a large Turán graph $T(m+1,n')$ [19]. Twenty years later, Erdős and Simonovits [17] deduced from this theorem a beautiful link between two seemingly disparate graphical parameters. For a graph G, let us denote by $\mathrm{ex}(n,G)$ the largest number of edges possible in a graph on n vertices which contains no copy of G, and by $\chi(G)$ the *chromatic number* of G, that is, the smallest number of colours with which the vertices can be painted so that no two adjacent vertices receive the same colour. The Erdős-Simonovits theorem states that

$$\lim_{n\to\infty}\frac{\mathrm{ex}(n,G)}{n^2}=\frac{1}{2}\left(\frac{\chi(G)-2}{\chi(G)-1}\right).$$

The Erdős-Stone theorem has been refined successively by Bollobás and Erdős [6], Bollobás, Erdős and Simonovits [7], and Chvátal and Szemerédi [9], the meaning of 'large' being sharpened each time. Very basic problems remain, however. For instance, when G is bipartite (that is, when $\chi(G)=2$), the Erdős- Simonovits theorem says only that $\mathrm{ex}(n,G)=o(n^2)$. Even in the very special case where G is a circuit of length $2k$, the order of magnitude of $\mathrm{ex}(n,G)$ is unknown; it is a long-standing conjecture of Erdős that $\mathrm{ex}(n,G)=O(n^{1+\frac{1}{k}})$.

Ramsey theory, too, has evolved into a full-fledged field under Erdős' influence. The germ of this theory is a theorem proved in 1930 by logician Frank Ramsey [25], asserting that for any partition of the r- subsets of an n-set into k classes, there is necessarily an m-subset of the n-set all of whose r-subsets belong to the same class, provided that n is sufficiently large; the smallest such value of n is known as the *Ramsey number.* Here, 'sufficiently large' means very large; even when $r = k = 2$, the Ramsey number $r(m)$ is greater than $2^{\frac{m}{2}}$, and the exact value of $r(5)$ has yet to be determined. Because of this, Ramsey's theorem has no practical significance. It has, however, proved to be a very useful theoretical tool, establishing a necessary modicum of order and structure in situations where none is apparent. Another such tool, one of enormous power, is Endre Szemerédi's 'regularity lemma' ([26], see also [22]). "The great Szemerédi", as Erdős impishly used to refer to him, wrote many papers with Paul, and the philosophy of the regularity lemma clearly owes much to his teaching. Szemerédi, and other Hungarian mathematicians too numerous to mention, have successfully carried Erdős' approach and methods over to fields as diverse as computational complexity and group theory, as well as number theory, set theory and classical analysis. The impact of Erdős on Hungarian mathematics is well documented in the richly detailed biographical article by László Babai [2].

Arguably one of Erdős' major achievements in combinatorics was the inception and development of the probabilistic method. He first used it, in 1947, to determine the lower bound for Ramsey numbers noted above [12], and at once appreciated the power and potential of this tool. He proceeded to apply it to many further questions which had resisted other approaches. In 1961, he used it with striking efficacy to establish the existence of graphs with arbitrarily large girth and chromatic number, thereby showing that the chromatic number is very much a global invariant [13]. And in 1981, with Siemion Fajtlowicz, he applied similar methods to demolish a 25-year-old conjecture of György Hajós, asserting that every graph of chromatic number m contains a subdivision (topological copy) of the complete graph on m vertices. A family of counterexamples had been found already by Paul Catlin [8]; what Erdős and Fajtlowicz [14] showed is that almost every graph is a counterexample. Now a standard proof technique in combinatorics [18, 1], the probabilistic method is used with ever-increasing frequency, as a glance at the literature will confirm.

It is often said that there are two types of mathematician — the problem solver and the theory builder. Erdős was both. He built theories by solving and posing problems, and by stimulating others to contribute to the cause. For him, mathematics was a communal enterprise. There was no place for the *prima donna*; what was of primary importance was the solution, not the solver. "Prove and conjecture" was his maxim. He shared his ideas freely, with anyone and everyone. This selfless approach to Science, combined with a manifest humanity, generosity and openness, encouraged many young — and not so young — mathematicians the world over to do battle with his questions and conjectures. He had (as Béla Bollobás observed in the biographical film "N is a Number" [11]) a remarkable talent for matching problems to people. (Statistics on Erdős' prodigious output, and also on his collaborations, can be found at *http://www.acs.oakland.edu/ grossman/erdoshp.html*, a home page maintained by Jerrold Grossman.)

While he commanded enormous respect, he was not in the least intimidating; indeed, he was eminently approachable. No doubt his sense of humour had something to do with this.

He had a story (often a true one) for every occasion, and made up not a few himself. Jingles, too, such as:

A theorem a day brings promotion and pay,
A theorem a year and you're out on your ear.

With irony, he would express incomprehension at the ways of the world and its rulers. He himself had his share of clashes with authority, notably when he was refused a reentry visa to the U.S. during the McCarthy era because of his ingenuous honesty in responding to an immigration officer's questions. In the 1970's, he remained in voluntary exile from Hungary for several years in protest at the government's denial of entry visas to a number of Israeli mathematicians. And earlier this year he made known his unhappiness with the actions of the University of Waterloo in Canada by renouncing the honorary degree it had awarded him. He would express disapproval or annoyance by uttering the oath "Fascism, Stalinism", occasionally concatenated with "Marxism, Capitalism, McCarthyism, ...". This was the case, for instance, when he made a poor shot at ping-pong. Once, during a lecture, a fly that persisted in bothering him despite several attempts at swatting was reproached with the words "Fascist monster".

Even God did not escape his irony. Erdős was evidently sceptical of the Almighty's presumed benevolence (if indeed he believed in Him at all), and referred to him as the "SF" (Supreme Fascist). Their relationship was ambiguous, however, for God was also the keeper of a Book in which were to be found the ideal proofs of all theorems. Notwithstanding Erdős' undisputed technical virtuosity, discovering a proof "from the Book" was the ultimate goal. His favourite example was L.M. Kelly's beautiful proof (see [10]) of the Sylvester-Gallai theorem, that every finite set of points in the Euclidean plane determines a two-point line: select a point p and a line ℓ whose distance is positive but as small as possible; the assumption that ℓ has at least three points yields an immediate contradiction.

Erdős began cultivating a special relationship with death and aging rather early on, as though to preempt the SF and thereby fend off the inevitable. This he did with typical humour, appending a string of initials to his name, starting with P.G.O.M. (Poor Great Old Man) and adding to these from the age of sixty onwards, every five years : L.D. (Living Dead), A.D. (Archaeological Discovery), L.D. (Legally Dead), C.D. (Counts Dead), N.D. (Nearly Dead). His prediction that the next one would be just plain D. was, regrettably, all too accurate. The rapport grew still stronger with the death in 1971 of his mother, to whom he was deeply attached. The two enemies were "old age and stupidity", and the best one could hope for was "an easy cure". This, at least, he was granted. The world of mathematics has lost a unique and extraordinary individual, and very many of us a true friend.

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