Permutations and Combinations

Definition 1: Permutation of a set of distinct objects is an ordered arrangement of these objects. (Order matters, no repetition).

Example: $S = \{1, 2, 3\}$, a permutation is (3,1,2).

Definition 2: $P(n, r)$ is a number of r-permutations of a set with n objects.

Theorem: Given S with n distinct elements, then:

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

Proof: given in class.

Definition 3: Combinations.
An r-combination of n elements is an unordered selection of r elements from n elements.

Example: $S = \{1, 2, 3, 4\}$ then $\{1, 2, 3\}$ is a 3-combination of S.

$$C(n, r) = \frac{n!}{r!(n-r)!}; 0 \leq r \leq n$$

$$P(n, r) = C(n, r) \cdot P(r, r)$$

$$C(n, r) = C(n, n-r)$$

Binomial coefficient: $C(n, r) = \binom{n}{r}$

Theorem: Pascal identity.

$$C(n+1, k) = C(n, k-1) + C(n, k)$$

Theorem: $\sum_{k=0}^{n} C(n,k) = 2^n$

Proof: given in class.

Vandermonde's Identity

Let m, n, and r be nonnegative integers, and $r \leq m, n$. Then:

$$C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k)$$

Proof: given in class.

The Binomial Theorem

Let x and y be variables, and n a positive integer. Then:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$= \sum_{j=0}^{n} C(n,j)x^{n-j}y^j.$$

Proof: by induction.

Example: $(x+y)^4 = \sum_{j=0}^{4} C(4,j)x^{4-j}y^j$
$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$
Solution given in class.

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$
Solution given in class.

Theorem: Let n be a positive integer. Then:

$$\sum_{k=0}^{n} (-1)^k C(n,k) = 0$$

Proof given in class.

Permutations with Repetition

Example: For $\{1, 2, 3, 4\}$, (2,2,1) and (2,2,3) are permutations with repetition allowed.

The number of r-permutations of n objects *with repetition* is:

$$n^r = \underbrace{n \cdot n \cdot \cdots \cdot n}_{r\ times}$$

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Combinations with Repetition

Theorem: There are $C(n+r-1, r)$ r-combinations from a set with n elements when repetition is allowed.

Example: How many ways to select 5 bills from a bag containing 1$, 2$, 5$, 10$, 20$, 50$, and 100$ bills? Assuming that the order in which the bills are chosen does not matter.

Example: How many solutions does the equation: $x_1 + x_2 + x_3 + x_4 = 11$ have, where x_1, x_2, x_3, x_4 are nonnegative integers?

Combinations and Permutations with or without Repetition

Type	Repetition Allowed?	Formula
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combitations	No	$\frac{n!}{r!(n-r)!}$
r-permutations	Yes	n^r
r-combitations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$

Note: $C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!}$

Permutations of Sets with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word $SUCCESS$?

Theorem: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ... , and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Distributing Objects into Boxes

Example: How many ways are there to distribute 5 cards to each of four players from a deck of 52 cards?

Theorem: The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, ..., k$ equals:

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$