Claude Berge: 5.6.1926 − 30.6.2002

Vašek Chvátal
Department of Computer Science
Rutgers University
Piscataway, NJ 08854, USA. *

The departure of Claude Berge leaves a painful void in many of our lives. He enchanted people around him with his multiple talents, his great erudition in diverse domains, his sense of humour, his modesty, his love of life.

Although primarily a combinatorist, Berge made a lasting mark on other subjects with his early mathematical work. His treatise on game theory [2] introduced an alternative to the Nash equilibrium, which has become known as the *Berge equilibrium* [1, 28, 32]. His book on topological spaces [4] introduced a theorem which has become known as the *Berge maximum theorem* and is considered one of the most useful tools in economic theory [31, 35, 39].

Up to the 1950's, many mathematicians considered combinatorics and graph theory somewhat disreputable. Berge did a lot to change this perception. His 1958 monograph on graph theory [3] was translated into English, Russian, Spanish, Romanian, and Chinese within five years. As Daniel Dugué [26] put it:

> *Si le mot "graphe" était à peine évoqué en France pour décrire une représentation sagitale, utilisée ponctuellement pour résoudre une récréation mathématique isolée, il fallut attendre Claude Berge pour s'apercevoir que ces théorèmes pouvaient être généralisés et former une véritable théorie mathématique au même titre que la Théorie des Ensembles; avec en outre des algorithmes permettant de résoudre des problèmes pratiques.*

Berge's book [13], written jointly with Ghouila-Houri on the subject of programming, games, and transportation networks, appeared in 1962 and was translated into English, German, Spanish, and Chinese by 1969. In the preface to the English translation of Berge's 1968 monograph on combinatorics [6], Gian-Carlo Rota wrote:

> *Two Frenchmen have played a major rôle in the renaissance of combinatorics: Berge and Schützenberger. Berge has been the more prolific writer, and his books have carried the word farther and more effectively that anyone anywhere. I recall the pleasure of reading the*

> *disparate examples in his first book, which made it impossible to for-*
> *get the material. Soon after reading, I would be one of many who*
> *unknotted themselves from the tentacles of the continuum and joined*
> *the Rebel Army of the Discrete.*

Berge's subsequent books [7, 10, 11] concern mostly generalizations of various aspect of graph theory to the theory of *hypergraphs*, a term coined by Berge himself; these differ from (undirected) graphs in that each of their "edges" may have an arbitrary number of vertices rather than just two.

Much of Claude Berge's research revolved around min-max formulas typified by the classic theorem proved independently by König and Hall:

> *in every bipartite graph, the smallest size of a vertex-cover (= a set*
> *of vertices that meets every edge) equals the largest size of a matching*
> *(= a set of pairwise disjoint edges).*

Such theorems are closely related to the duality principle of linear programming. This principle guarantees that, for every matrix A in $\mathbf{R}^{m \times n}$ (and with e standing for the all-ones vectors), we have

$$\min\{e^T x : x \in \mathbf{R}^n, Ax \geq e, x \geq 0\} = \max\{y^T e : y \in \mathbf{R}^m, y^T A \leq e, y \geq 0\};$$

the König-Hall theorem asserts that, as long as A is the edge-vertex incidence matrix of a bipartite graph (meaning that the n columns of A are indexed by the n vertices and that its m rows of A are the characteristic vectors of the m edges), the left-hand side minimum is attained by a vector x in $\{0,1\}^n$ and the right-hand side maximum is attained by a vector y in $\{0,1\}^m$. Berge [9] proved that this stronger conclusion holds for a much wider class of matrices, which he named *balanced*: these are zero-one matrices with no square submatrix of odd order and with precisely two 1's in each row and each column. (With his penchant for hypergraphs, Berge [8] considered the rows of these matrices as incidence vectors of hypergraph edges.) In fact, he showed that, as long as A is balanced, both polyhedra

$$\{x \in \mathbf{R}^n : Ax \geq e, x \geq 0\} \quad \text{and} \quad \{y \in \mathbf{R}^m : y^T A \leq e, y \geq 0\};$$

have only integral extreme points.

An earlier notion introduced by Berge and also related to the König-Hall theorem starts out with the trivial inequality $\chi(G) \geq \omega(G)$, where $\chi(G)$ denotes the *chromatic number* of a graph G (meaning the smallest number of colors that suffice to color the vertices in such a way that every two adjacent vertices receive distinct colors) and $\omega(G)$ denotes the *clique number* of a graph G (meaning the largest number of pairwise adjacent vertices). Berge proposed studying the class of graphs G such that every *induced subgraph* F of G (meaning a subgraph of G defined just by its set W of vertices and including all the edges of G that have both endpoints in W) satisfies the min-max equality $\chi(F) = \omega(F)$; nowadays, such graphs are called *perfect*. (Every bipartite graph B yields a graph G that is perfect by virtue of the König-Hall theorem: vertices of G are the edges of B

and two vertices of G are adjacent if and only if, as edges of B, they are disjoint.)
Insisting on the equality $\chi = \omega$ not just for the graph itself, but also for all its
induced subgraphs might seem contrived, but it turned out to be wonderfully
inspired. Whereas the class of graphs G with $\chi(G) = \omega(G)$ is unintersting
(every graph is an induced subgraph of such a graph and recognizing graphs
for which $\chi = \omega$ is polynomially equivalent to the notoriously difficult problem
of recognizing graphs for which $\chi \leq 3$), the class of perfect graphs has a most
natural characterization in terms of the clique-vertex incidence matrix (whose
columns are indexed by the n vertices and whose rows are the characteristic
vectors of cliques): as pointed out by Chvátal [16], results of Lovász [33, 34]
imply that a graph with clique-vertex incidence matrix A is perfect if and only
if the polyhedron

$$\{x \in \mathbf{R}^n : Ax \leq e, x \geq 0\}$$

has only integral extreme points.

Perfect graphs have proved to be one of the most stimulating and fruitful
concepts of modern graph theory: there are now three books [29, 12, 36] and
nearly six hundred papers [18] on the subject and the 2000 Mathematics Subject
Classification assigns perfect graphs their own code, 05C17. The origin of this
development was Berge's conjecture that

> *a graph is perfect if and only if neither it nor its complement contains*
> *a chordless cycle whose length is odd and at least five.*

Berge publicized this conjecture first in April 1960 in a lecture at an interna-
tional meeting on graph theory organized by Horst Sachs at the Martin Luther
University, Halle-Wittenberg; he only published it three years later [5]. This
conjecture became known as the Strong Perfect Graph Conjecture; the term
Weak Perfect Graph Conjecture was reserved for its corollary,

> *the complement of a perfect graph is perfect,*

proved in 1971 by Lovász [33]. Another milestone in the evolution of perfect
graph theory was the 1981 Grötschel-Lovász-Schrijver polynomial-time algo-
rithm for finding, in a perfect graph G, a clique of size $\omega(G)$ and a colouring by
$\chi(G)$ colours [30].

There are theorems that elucidate the structure of objects in some class $\mathbf{C}$ by
showing that every object in $\mathbf{C}$ has either a prescribed and relatively transparent
structure or one of a number of prescribed structural faults, along which it can
be decomposed. An early example is the Kronecker Decomposition Theorem
for Abelian groups; a celebrated example in combinatorics is Paul Seymour's
decomposition theorem for regular matroids [38]. Berge's notions of balanced
matrices and perfect graphs have been treated this way. Conforti, Cornuéjols,
and Rao [22] proved that every balanced matrix is either totally unimodular
(and therefore decomposable in its own right by virtue of Seymour's theorem)
or has a structural fault called a double star cutset. Following Burlet and Uhry's
work on parity graphs [15], many people [14, 20, 21, 27, 37, 19] tried to apply
this paradigm to *Berge graphs*, meaning graphs G such that neither G nor its

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complement contains a chordless cycle whose length is odd and at least five. What has eventually emerged are four classes of basic Berge graphs and three kinds of structural faults. The four basic classes are bipartite graphs, their complements, line-graphs of bipartite graphs, and their complements; the three kinds of structural faults are skew partitions [17], 2-joins [25], and 2-joins in the complement. In February 2001, Conforti, Cornuéjols, and Vušković [23] proved that Berge graphs without chordless cycles of length four either belong to one of the four basic classes or have one of the three structural faults (with skew partition restricted to its special case, a star-cutset). In September 2001, Seymour organized a workshop in Princeton, where the objective of proving that every Berge graph either belongs to one of the four basic classes or has one of the three structural faults was highlighted. In a remarkable sequence of results, by Chudnovsky, Robertson, Seymour, and Thomas, by Conforti, Cornuéjols, and Zambelli [24], and with the final decisive push in May 2002 by Chudnovsky and Seymour alone, this objective was accomplished. (The proof is long and difficult; its details are still being checked.) Since the four basic classes of Berge graphs are known to be perfect and since no minimal imperfect Berge graph has any of the three structural faults, the Strong Perfect Graph Conjecture follows. Perfect graphs had come of age just in time for their creator to witness the rite of passage.

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