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Finite Sholander trees, trees, and their betweenness

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ABSTRACT

We provide a proof of Sholander's claim [M. Sholander, Trees, lattices, order, and betweenness, Proceedings of the American Mathematical Society 3 (1952) 369–381] concerning the representability of collections of so-called segments by trees, which yields a characterization of the interval function of a tree. Furthermore, we streamline Burigana's characterization [L. Burigana, Tree representations of betweenness relations defined by intersection and inclusion, Mathematics and Social Sciences 185 (2009) 5–36] of tree betweenness and provide a relatively short proof.

1. Introduction

Trees form one of the most simple yet important classes of graphs with countless applications ranging from data structures and very-large-scale integration (VLSI) design over mathematical psychology to gardening. Here, we consider two closely related papers on trees: one by Sholander [19], published in 1952, and the other by Burigana [4], published in 2009. Both of these papers include characterizations of certain ternary relations associated with trees.

We use the term *tree* in the sense defined by Kőnig [11, p. 47]: a finite, simple, undirected, and connected graph without cycles. Sholander [19] used this term in a different sense: he studied collections of so-called *segments*, which are subsets of a set V indexed by all ordered pairs of elements of V, and he referred to such a collection as a *tree* if it satisfies certain postulates. He stated without a proof that these postulates characterize the function that assigns to every pair of vertices of a tree in the sense of Kőnig the set of vertices on the path joining these two vertices; nowadays, this function is called the *interval function* of the tree [13]. Interval functions of Kőnig trees are easily seen to be trees in Sholander's sense, but it is not obvious that all finite Sholander trees are representable as interval functions of Kőnig trees. In Section 2, we supply the missing proof of this claim.

The *tree betweenness* of a tree T is defined as the set of all ordered triples (x, y, z) such that x, y, z are (not necessarily distinct) vertices of T and y belongs to the path in T that joins x and z; the *strict tree betweenness* of T is defined as the set of all ordered triples (x, y, z) such that x, y, z are pairwise distinct vertices of T and y belongs to the path in T that joins x and z. It is a routine matter to restate Sholander's characterization of the interval function of a tree as a characterization of tree betweenness; this was done, with refinements, by Sholander himself in the same paper [19]; subsequently, Defays [6] found another characterization of tree betweenness. Burigana (Theorem 1 in [4]) characterized strict tree betweenness by a list of five properties that do not involve the notion of a tree. His proof is spread over some seven pages; in Section 3, we give

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a shorter proof; actually, we prove a simpler theorem, of which Burigana's is an instant corollary. In addition, we restate the simplified characterization of strict tree betweenness in terms of tree betweenness.

Before proceeding to our results, let us put their subject in a broader context by mentioning a few related references. Mulder and Nebeský [13–15] studied interval functions of arbitrary graphs. Tree betweenness is a special kind of *metric betweenness* that, for a prescribed metric space (V, dist), consists of all ordered triples (x, y, z) such that x, y, z are (not necessarily distinct) points of V and $\mathrm{dist}(x, y) + \mathrm{dist}(y, z) = \mathrm{dist}(x, z)$. This concept was first studied by Menger [12] in 1928; references to subsequent work on it can be found in [5]. Another special kind of metric betweenness is *Euclidean betweenness*, where the metric space is a Euclidean space or some subspace of it. In his development of geometry, Euclid used the notion of betweenness only implicitly; its explicit axiomatization was first carried out by Pasch [16] and then gradually refined by Peano [17], Hilbert [8], Veblen [20], and Huntington and Kline [9]. In particular, Huntington and Kline suggested the study of other ternary relations (meaning subsets of V^3, where V is some set) that resemble Euclidean betweenness: for example, they mention the set of all ordered triples (x, y, z) such that x, y, z are natural numbers and $y = xz$. Pitcher and Smiley [18] continued in this direction. Another particular kind of betweenness is *order betweenness* that, for a prescribed partially ordered set $(V, \preceq)$, consists of all ordered triples (x, y, z) such that x, y, z are (not necessarily distinct) points of V and $x \preceq y \preceq z$ or $z \preceq y \preceq x$. This concept was first studied by Birkhoff [3] in 1948; Altwegg [2] characterized order betweenness by a list of six properties that do not involve the notion of a partially ordered set; subsequently, Sholander [19] and Düntsch and Urquhart [7] found other characterizations of order betweenness.

2. Finite Sholander trees are trees

Sholander studies mappings that assign to each ordered pair (a, b) of elements of a set V a subset of V, which we denote as $[ab]$. From postulates

(S) $\forall a, b, c \in V : \exists d \in V : [ab] \cap [bc] = [bd]$,
(T) $\forall a, b, c \in V : [ab] \subseteq [ac] \Rightarrow [ab] \cap [bc] = \{b\}$,

he derives a number of corollaries that include

(1.2) $\forall a, b \in V : b \in [ab]$,
(1.4) $\forall a, b \in V : [ab] = [ba]$,
(1.5) $\forall a, b, c \in V : b \in [ac] \Leftrightarrow [ab] \subseteq [ac]$,
(1.7) $\forall a, b, c \in V : (b \in [ac] \wedge c \in [ab]) \Rightarrow b = c$,
(1.10) $\forall a, b, c, d \in V : [ab] \cap [bc] = [bd] \Rightarrow [ad] \cap [dc] = \{d\}$.

(The labels (S), (T), (1.2), etc., used in this section are copied directly from [19] for ease of reference.) Then he defines a tree as a mapping $(u, v) \mapsto [uv]$ from V^2 to 2^V that satisfies (S), (T), and

(U_1) $\forall a, b, c \in V : [ab] \cap [bc] = \{b\} \Rightarrow [ab] \cup [bc] = [ac]$.

Having noted that Kőnig defined a tree as a finite connected graph that contains no cycles, he states [19, p. 370] that "*Trees in our sense which are finite are trees in Kőnig's sense.*". In formalizing this statement, we let $[uv]_T$ denote the set of all vertices on the path in a tree T that joins a vertex u and a vertex v.

Theorem 1. *Let V be a finite set. A mapping $(u, v) \mapsto [uv]$ from V^2 to 2^V satisfies* (S), (T), *and* (U_1) *if and only if there is a tree T with vertex set V such that $[vw]_T = [vw]$ for all pairs v, w of its vertices.*

Sholander does not prove this theorem, but goes on to derive from the conjunction of (S), (T), and (U_1) a number of corollaries that include

(2.1) $\forall a, b, c \in V : b \in [a, c] \Leftrightarrow [a, b] \cap [b, c] = \{b\} \Leftrightarrow [a, b] \cup [b, c] = [a, c]$,
(5.2) $\forall a, b, x, y \in V : x, y \in [a, b] \Rightarrow (x \in [a, y] \wedge y \in [x, b]) \vee (y \in [a, x] \wedge x \in [y, b])$.

We are going to derive Theorem 1 from Sholander's results.

The following fact is well known (for instance, Exercise 12 in Section 2.3., p. 314, of [10], and the answer on p. 558). We give its straightforward proof just for the sake of completeness.

Lemma 2. *Let V be a finite set, and let r be an element of V. If $\preceq$ is a partial order on V such that*

(i) $\forall w \in V : r \preceq w$,
(ii) $\forall u, v, w \in V : (u \preceq w \wedge v \preceq w) \Rightarrow (u \preceq v \vee v \preceq u)$,

then there is a tree T with vertex set V such that $u \preceq x \Leftrightarrow u \in [rx]_T$.

Proof. The proof is carried out by induction on $|V|$. If $|V| = 1$, then T consists of a single vertex. If $|V| > 1$, then enumerate the minimal elements of $V \setminus \{r\}$ as $r_1, r_2, \ldots, r_k$, and set $V_i = \{x \in V : r_i \preceq x\}$. Property (ii) guarantees that the sets $V_1, V_2, \ldots, V_k$ form a partition of $V \setminus \{r\}$. By the induction hypothesis, there are trees $T_1, T_2, \ldots, T_k$ such that each T_i has V_i for its vertex set and such that elements u, x of V_i satisfy $u \preceq x$ if and only if u is on the path from r_i to x in T_i. The union of $T_1, T_2, \ldots, T_k$ along with vertex r and the k edges $rr_1, rr_2, \ldots, rr_k$ has the property required of T. □

Proof of Theorem 1. The "if" part is clear. To prove the "only if" part, choose an arbitrary element of V, call it r, and write $u \preceq x$ if and only if $u \in [rx]$. This binary relation is a partial order: (1.2) with $a = r$ means that $\preceq$ is reflexive, (1.7) with $a = r$ means that $\preceq$ is antisymmetric, and (1.5) with $a = r$ implies that $\preceq$ is transitive. By (1.2) and (1.4) with $b = r$, $a = x$, this partial order has property (i) of Lemma 2; by (5.2) with $a = r$, $b = w$, $x = u$, $y = v$, it has property (ii) of Lemma 2. This lemma guarantees that there is a tree T with vertex set V such that

(α) $[rx]_T = [rx]$ for all vertices x of T.

We will prove that this T has the property specified in the theorem. To begin, let us generalize (α) to

(β) if $u \in [rx]_T$, then $[ux]_T = [ux]$.

To verify this, note that (2.1) with $a = r$, $b = u$, and $c = x$ implies that $[ru] \cap [ux] = \{u\}$ and $[ru] \cup [ux] = [rx]$, and so $[ux] = ([rx] \setminus [ru]) \cup \{u\} = ([rx]_T \setminus [ru]_T) \cup \{u\} = [ux]_T$. The conclusion of the theorem is a generalization of (β):

(γ) $[vw]_T = [vw]$ for all pairs v, w of vertices of T.

To verify (γ), consider an arbitrary pair v, w of vertices of T. Since T contains no cycle, there is a vertex u such that $[rv]_T \cap [rw]_T = [ru]_T$ and $[vw]_T = [vu]_T \cup [uw]_T$. By (1.4), we have $[vr] \cap [rw] = [rv] \cap [rw] = [rv]_T \cap [rw]_T = [ru]_T = [ru]$, and so (1.10) with $a = v$, $b = r$, $c = w$, and $d = u$ guarantees that $[vu] \cap [uw] = \{u\}$. Now (2.1) with $a = v$, $b = u$, and $c = w$ implies that $[vu] \cup [uw] = [vw]$; using (β) and (1.4), we conclude that $[vw]_T = [vu]_T \cup [uw]_T = [uv]_T \cup [uw]_T = [uv] \cup [uw] = [vu] \cup [uw] = [vw]$. □

Our proofs of Lemma 2 and Theorem 1 yield an efficient way of reconstructing a tree from its collection of segments $[uv]$. Of course, the simplest way of doing that is to make distinct u and v adjacent if and only if $[uv] = \{u, v\}$.

3. Strict tree betweenness and tree betweenness

A *ternary relation* on a set V means a subset of V^3; a ternary relation $\mathcal{B}$ is called *strict* if $(x, y, z) \in \mathcal{B}$ implies that x, y, and z are pairwise distinct. Given a ternary relation $\mathcal{B}$ on a set V, we follow Burigana [4] in writing $N(u, v, w)$ to mean that u, v, and w are pairwise distinct elements of V, and $(u, v, w) \notin \mathcal{B}$, $(v, w, u) \notin \mathcal{B}$, $(w, u, v) \notin \mathcal{B}$.

Theorem 3. *Let V be a finite set. A strict ternary relation $\mathcal{B}$ on V is a strict tree betweenness if and only if it satisfies*

(S_1) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (w, v, u) \in \mathcal{B}$,
(S_2) $\forall u, v, w, z \in V : (u, v, w), (v, w, z) \in \mathcal{B} \Rightarrow (u, w, z) \in \mathcal{B}$,
(S_3) $\forall u, v, w, z \in V : (u, v, w), (u, w, z) \in \mathcal{B} \Rightarrow (v, w, z) \in \mathcal{B}$,
(S_4) $\forall u, v, w \in V : N(u, v, w) \Rightarrow \exists c \in V : (u, c, v), (u, c, w) \in \mathcal{B}$.

Proof. The "only if" part is clear. To prove the "if" part, we first derive from (S_1)–(S_4) a few corollaries:

(S_5) $\forall u, v, w, z \in V : (u, v, w), (u, w, z) \in \mathcal{B} \Rightarrow (u, v, z) \in \mathcal{B}$,
(S_6) $\forall u, v, w, z \in V : (u, v, z), (v, w, z) \in \mathcal{B} \Rightarrow (u, v, w), (u, w, z) \in \mathcal{B}$,
(S_7) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (v, u, w) \notin \mathcal{B}$,
(S_8) $\forall u, v, w, z \in V : (u, v, z), (u, w, z) \in \mathcal{B} \Rightarrow v = w \vee (u, v, w) \in \mathcal{B} \vee (u, w, v) \in \mathcal{B}$,
(S_9) $\forall u, v, w, z \in V : (u, v, z), (u, w, z) \in \mathcal{B} \Rightarrow v = w \vee (w, v, u) \in \mathcal{B} \vee (w, v, z) \in \mathcal{B}$,
(S_{10}) $\forall r, u, x, y, z \in V : (r, u, x), (r, u, z), (x, y, z) \in \mathcal{B} \Rightarrow y = u \vee (r, u, y) \in \mathcal{B}$.

In these derivations, we will invoke (S_1) only tacitly whenever we use it. (Whenever we invoke *reversed* (S_i) we mean that we invoke the conjunction of (S_1) and (S_i).)

Property (S_5) comes directly out of (S_3) followed by (S_2). Property (S_6) comes directly out of reversed (S_3) followed by (S_2). To derive (S_7), note that $(w, v, w) \notin \mathcal{B}$ as $\mathcal{B}$ is strict and that $(u, v, w) \in \mathcal{B}, (w, v, w) \notin \mathcal{B}$ implies that $(w, u, v) \notin \mathcal{B}$ by (S_2).

We will derive (S_8) and (S_9) along the lines of Burigana's proof [4] of his Lemma 1(i). Similar properties were considered by Adeleke and Neumann in [1].

To derive (S_8), assume the contrary: $(u, v, z), (u, w, z) \in \mathcal{B}$ but $(u, v, w) \notin \mathcal{B}, (u, w, v) \notin \mathcal{B}$ for some u, v, w, z in V such that $v \neq w$. From $(u, v, z) \in \mathcal{B}$, we get $(v, u, z) \notin \mathcal{B}$ by (S_7); in turn, from $(z, w, u) \in \mathcal{B}$ and $(z, u, v) \notin \mathcal{B}$, we get $(w, u, v) \notin \mathcal{B}$ by (S_2). Now $N(u, v, w)$, and so two different applications of (S_4) give points c and d such that $(w, c, u), (w, c, v) \in \mathcal{B}$ and $(v, d, u), (v, d, w) \in \mathcal{B}$. From $(u, c, w) \in \mathcal{B}$ and $(u, w, z) \in \mathcal{B}$, we get $(c, w, z) \in \mathcal{B}$ by (S_3); in turn, from $(v, c, w) \in \mathcal{B}$ and $(c, w, z) \in \mathcal{B}$, we get $(v, w, z) \in \mathcal{B}$ by (S_2). Similarly, from $(u, d, v) \in \mathcal{B}$ and $(u, v, z) \in \mathcal{B}$, we get $(d, v, z) \in \mathcal{B}$ by (S_3); in turn, from $(w, d, v) \in \mathcal{B}$ and $(d, v, z) \in \mathcal{B}$, we get $(w, v, z) \in \mathcal{B}$ by (S_2). But then (S_7) is contradicted by $(v, w, z), (w, v, z) \in \mathcal{B}$.

Property (S_9) comes out of (S_8) followed by (S_3) with v and w switched.

To derive (S_{10}), assume that $(r, u, x), (r, u, z), (x, y, z) \in \mathcal{B}$, and write $a \prec b$ if and only if $(r, a, b) \in \mathcal{B}$. This binary relation is a strict partial order: it is irreflexive since $\mathcal{B}$ is strict and it is transitive by (S_5). By assumption, the set $\{v : v \prec x, v \prec z\}$ is nonempty; consider any of its maximal elements and denote it w. By (S_8) and by maximality of w, we have $w = u$ or $u \prec w$, and so (S_5) reduces proving $y = u \vee (r, u, y) \in \mathcal{B}$ to proving $y = w \vee (r, w, y) \in \mathcal{B}$. By maximality of w, no c with $w \prec c$ satisfies $c \prec x, c \prec z$; from reversed (S_5), it follows that no c satisfies $(w, c, x), (w, c, z) \in \mathcal{B}$; since $\mathcal{B}$ is strict, w, x, z are pairwise distinct; now (S_4) implies that at least one of (w, x, z), (x, z, w), (z, w, x) belongs to $\mathcal{B}$. Interchangeability of x and z allows us to assume that at least one of (w, x, z), (z, w, x) belongs to $\mathcal{B}$. When $(w, x, z) \in \mathcal{B}$, we get first $(w, x, y) \in \mathcal{B}$ by (S_6) and then $(r, w, y) \in \mathcal{B}$ by reversed (S_2). When $(z, w, x) \in \mathcal{B}$, property (S_9) guarantees that $y = w$ or $(w, y, x) \in \mathcal{B}$ or $(w, y, z) \in \mathcal{B}$; if $(w, y, x) \in \mathcal{B}$ or $(w, y, z) \in \mathcal{B}$, then $(r, w, y) \in \mathcal{B}$ by reversed (S_3).

Now (S_5)–(S_{10}) are established, and we proceed to prove the "if" part of the theorem by induction on $|V|$. If $|V| = 1$, then the statement is trivial. If $|V| > 1$, then we choose an arbitrary element of V, call it r, and write $a \prec b$ if and only if $(r, a, b) \in \mathcal{B}$. This binary relation is a strict partial order: it is irreflexive since $\mathcal{B}$ is strict and it is transitive by (S_5). Enumerate the minimal elements of $V \setminus \{r\}$ as $r_1, r_2, \ldots, r_k$, and set $V_i = \{r_i\} \cup \{b \in V : r_i \prec b\}$. Property (S_8) guarantees that the sets $V_1, V_2, \ldots, V_k$ form a partition of $V \setminus \{r\}$. By the induction hypothesis, there are trees $T_1, T_2, \ldots, T_k$ such that each T_i has V_i for its vertex set and such that elements x, y, z of each V_i satisfy $(x, y, z) \in \mathcal{B}$ if and only if y is an internal vertex of the path in T_i that joins x and z. Let T denote the union of $T_1, T_2, \ldots, T_k$ along with vertex r and the k edges $rr_1, rr_2, \ldots, rr_k$. We claim that elements x, y, z of V satisfy $(x, y, z) \in \mathcal{B}$ if and only if y is an internal vertex of the path in T that joins x and z. Interchangeability of x and z allows us to distinguish between three cases:

Case 1: *$x, z \in V_i$ for some i.* In this case, the path P in T that joins x and z is a path in T_i. If y is an internal vertex of P, then $y \in V_i$, and so the induction hypothesis guarantees that $(x, y, z) \in \mathcal{B}$; conversely, if $(x, y, z) \in \mathcal{B}$, then $y \in V_i$ (by reversed (S_3) if r_i is one of x, z and by (S_{10}) otherwise), and so the induction hypothesis guarantees that y is an internal vertex of P.

Case 2: *$x = r, z \in V_i$ for some i.*

Subcase 2.1: $z = r_i$. In this subcase, the path in T that joins x and z consists of a single edge, and so it has no internal vertex. Minimality of r_i guarantees that there is no y such that $(x, y, z) \in \mathcal{B}$.

Subcase 2.1: $z \neq r_i$. In this subcase, y is an internal vertex of the path in T that joins x and z if and only if $y = r_i$ or y is an internal vertex of the path in T_i that joins r_i and z; our analysis of Case 1 shows that this occurs if and only if $y = r_i$ or $(r_i, y, z) \in \mathcal{B}$; reversed property (S_5) guarantees that $y = r_i \vee (r_i, y, z) \in \mathcal{B}$ implies that $(x, y, z) \in \mathcal{B}$; property (S_9) combined with the minimality of r_i guarantees that $(x, y, z) \in \mathcal{B}$ implies that $y = r_i \vee (r_i, y, z) \in \mathcal{B}$.

Case 3: *$x \in V_i, z \in V_j$ for some distinct i and j.* In this case, we claim that $(x, r, z) \in \mathcal{B}$; to justify this claim, let us assume the contrary. Since $x \in V_i$ and $(r, r_i, z) \notin \mathcal{B}$, property (S_5) implies that $(r, x, z) \notin \mathcal{B}$; similarly, since $z \in V_j$ and $(r, r_j, x) \notin \mathcal{B}$, property (S_5) implies that $(r, z, x) \notin \mathcal{B}$; now (S_4) gives a c such that $(r, c, x) \in \mathcal{B}, (r, c, z) \in \mathcal{B}$. Since $z \notin V_i$, we have $(r, r_i, z) \notin \mathcal{B}$; in particular, $c \neq r_i$. Since $(r, c, z) \in \mathcal{B}$ and $(r, r_i, z) \notin \mathcal{B}$, we have $(r, r_i, c) \notin \mathcal{B}$ by (S_5). Since $(r, c, x) \in \mathcal{B}$, minimality of r_i implies that $x \neq r_i$; in turn, $x \in V_i$ implies that $(r, r_i, x) \in \mathcal{B}$. Now $(r, c, x) \in \mathcal{B}, (r, r_i, x) \in \mathcal{B}, c \neq r_i, (r, r_i, c) \notin \mathcal{B}$, and so (S_8) implies that $(r, c, r_i) \in \mathcal{B}$, contradicting minimality of r_i. This contradiction proves that $(x, r, z) \in \mathcal{B}$.

A vertex y is an internal vertex of the path in T that joins x and z if and only if $y = r$ or y is an internal vertex of the path in T_i that joins x and r or y is an internal vertex of the path in T_j that joins r and z; our analysis of Case 2 shows that this occurs if and only if $y = r$ or $(x, y, r) \in \mathcal{B}$ or $(r, y, z) \in \mathcal{B}$; property (S_5) and its reversal guarantee that $y = r \vee (x, y, r) \in \mathcal{B} \vee (r, y, z) \in \mathcal{B}$ implies that $(x, y, z) \in \mathcal{B}$; property (S_9) guarantees that $(x, y, z) \in \mathcal{B}$ implies that $y = r \vee (x, y, r) \in \mathcal{B} \vee (r, y, z) \in \mathcal{B}$. □

Our proof of Theorem 3 yields an efficient way of reconstructing a tree from its strict betweenness $\mathcal{B}$. Of course, the simplest way of doing that is to make distinct u and w adjacent if and only if no v satisfies $(u, v, w) \in \mathcal{B}$.

Corollary 4 (*Burigana [4]*). *Let V be a finite set. A strict ternary relation $\mathcal{B}$ on V is a strict tree betweenness if and only if it satisfies*

- $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (w, v, u) \in \mathcal{B},$
- $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (v, u, w) \notin \mathcal{B},$
- $\forall u, v, w, z \in V : (u, v, w), (v, w, z) \in \mathcal{B} \Rightarrow (u, w, z) \in \mathcal{B},$
- $\forall u, v, w, z \in V : (u, v, w), (u, w, z) \in \mathcal{B} \Rightarrow (v, w, z) \in \mathcal{B},$
- $\forall u, v, w \in V : N(u, v, w) \Rightarrow \exists c \in V : (u, c, v), (u, c, w), (v, c, w) \in \mathcal{B}.$ □

Clearly, a ternary relation $\mathcal{C}$ on a finite set V is a tree betweenness if and only if it is the union of ternary relations $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ consists of all triples (u, v, w) in V^3 that satisfy $u = v$ or $v = w$ (or both) and $\mathcal{B}$ is the strict tree betweenness of a tree with vertex set V. This observation enables us to translate our characterization of strict tree betweenness into a characterization of tree betweenness.

Corollary 5. *Let V be a finite set. A ternary relation $\mathcal{C}$ on V is a tree betweenness if and only if it satisfies*

(T_1) $\forall u, v, w \in V : (u, v, w) \in \mathcal{C} \Rightarrow (w, v, u) \in \mathcal{C},$
(T_2) $\forall u, v, w, z \in V : (u, v, w), (v, w, z) \in \mathcal{C}, v \neq w \Rightarrow (u, w, z) \in \mathcal{C},$
(T_3) $\forall u, v, w, z \in V : (u, v, w), (u, w, z) \in \mathcal{C} \Rightarrow (v, w, z) \in \mathcal{C},$
(T_4) $\forall u, v, w \in V : N(u, v, w) \Rightarrow \exists c \in V : c \neq u$ *and* $(u, c, v), (u, c, w) \in \mathcal{C}.$
(T_5) $\forall u, v, w \in V : (u, v, w), (v, u, w) \in \mathcal{C} \Leftrightarrow u = v.$

Proof. The "only if" part is clear. To prove the "if" part, assume that $\mathcal{C}$ satisfies (T_1)–(T_5), let $\mathcal{B}$ denote the set of all triples (u, v, w) in $\mathcal{C}$ such that u, v, w are pairwise distinct, and set $\mathcal{A} = \mathcal{C} \setminus \mathcal{B}$. Clearly, $\mathcal{B}$ satisfies (S_1)–(S_4), and so it is a strict tree betweenness. By (T_5), all triples (u, v, w) in V^3 that satisfy $u = v$ belong to $\mathcal{A}$; in turn, by (T_1), all triples (u, v, w) in V^3 that satisfy $v = w$ belong to $\mathcal{A}$; now (T_5) guarantees that all triples (u, v, u) in $\mathcal{A}$ satisfy $v = u$. □

None of the four conditions (S_1)–(S_4) of Theorem 3 is implied by the conjunction of the other three, and none of the five conditions (T_1)–(T_5) of Corollary 5 is implied by the conjunction of the other four. To verify this, consider $V = \{u, v, w, z\}$ and the following five ternary relations on V:

$$\mathcal{B}_1 = \{(u, v, w), (u, v, z), (u, w, z), (v, w, z)\},$$
$$\mathcal{B}_2 = \{(u, v, w), (v, w, z), (w, z, u), (z, u, v), (w, v, u), (z, w, v), (u, z, w), (v, u, z)\},$$
$$\mathcal{B}_3 = \{(u, v, w), (u, v, z), (u, w, z), (w, v, z), (w, v, u), (z, v, u), (z, w, u), (z, v, w)\},$$
$$\mathcal{B}_4 = \{(u, z, v), (u, z, w), (v, z, w), (v, z, u), (w, z, u), (w, z, v)\},$$
$$\mathcal{B}_5 = V^3.$$

For each $i = 1, 2, 3, 4$, relation $\mathcal{B}_i$ satisfies all nine conditions except (S_i) and (T_i); relation $\mathcal{B}_5$ satisfies all nine conditions except (T_5).

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